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Felix Klein
Arnold Sommerfeld

The Theory of the Top
Volume III

Perturbations. Astronomical and Geophysical Applications

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Foreword to Volume III by Michael Eckert

Birkhäuser

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Foreword

When the second volume of the *Theory of the Top* was published in 1898, the mathematical foundation of the work was complete. Sommerfeld's text had already reached the size of a treatise of 512 pages—far more than Klein had initially expected. Applications were reserved for a concluding third volume, as both Klein and Sommerfeld had originally planned. Their expectations, however, were defied by the contingencies of life. Five years lapsed before Sommerfeld finished the third volume, and seven more years would pass before the appearance of the fourth and final volume.

A year after the publication of the second volume, Klein expressed in a letter to Sommerfeld a "quiet concern" about the pace of progress with "our top" [Klein 1899], which largely became Sommerfeld's top as the projected content expanded far beyond the bounds of Klein's original lectures. But Sommerfeld's pending move from Clausthal to Aachen, where he was called as professor of mechanics, left little room for additional work. Furthermore, Klein had involved Sommerfeld by this time with the editing of the physics volumes of the *Enzyklopädie der mathematischen Wissenschaften*, a long-term effort that would occupy Sommerfeld for almost three decades. Progress with Volume III of the *Theory of the Top* was confined to the gathering of pertinent material and correspondence with colleagues who were involved with practical applications. The initial momentum was waning. Occasional meetings with Klein and the publisher, however, roused Sommerfeld's sense of duty. "The top makes progress," he assured Klein early in 1902, "even though with interruptions" [Sommerfeld 1902a]. In June of 1902, he promised more progress for the holidays after the summer semester [Sommerfeld 1902b].

The third volume was intended to confront theory with practice, to account for deviations (perturbations or "disturbing influences," as they were called in the subtitle of this volume) from the abstract theory of Volumes I and II. As professor of mechanics at the *Technische Hochschule* in Aachen, the gap between theory and practice was an obvious concern for Sommerfeld. While gathering relevant material for the theory of the top, he published, for example, an article on the theory of railway brakes [Sommerfeld 1902c]. It is not accidental, therefore, that he dedicated the first chapter of Volume III to the influence of friction on the motion of the top. He was at pains, however, to preserve the actual technical applications for later, since he first wished to discuss more general issues, such as the numerical evaluation of integrals, that were involved in the problems under consideration. The second chapter of Volume III was devoted to astronomical and geophysical applications. The two chapters turned out to be enough for one volume, so that "the technical and physical applications," as Sommerfeld wrote in the preface to Volume III, "remain for the fourth (final) volume."

In the summer of 1902, the astrophysicist Karl Schwarzschild joined the project as an advisor for the astronomical applications—presumably at the request of Klein in order to advance the pace of progress. Schwarzschild had been called to Göttingen in 1901 as the new director of the astronomical observatory. In July 1902, Sommerfeld sent him a preliminary manuscript for review. "The manuscript must, in order to be consistent, be written by me. If you feel like rewriting one part or another, I would be more than happy, but the final authority for the manuscript before publication must be left to me" [Sommerfeld 1902d]. Thus Sommerfeld prevented any aspirations of co-authorship in this effort. The part of the chapter that dealt with astronomical applications took shape during the following months. Schwarzschild did not resent that Sommerfeld had not offered him joint authorship. In August 1902, Sommerfeld sent him a new draft and asked for a "most severe critique," because he felt "not at all at home" in this matter. "The more you add, eliminate, change, the better" [Sommerfeld 1902e]. In September 1902, Sommerfeld traveled to Göttingen in order to discuss some details of this chapter with Schwarzschild. By January 1903, "the fruit of our former deliberations" was in print, as Sommerfeld informed Schwarzschild, requesting some final corrections of one or another numerical value [Sommerfeld 1903]. In the beginning, they had addressed each other in a rather formal manner as *"Lieber Herr College"*; now they changed to the more customary *"Lieber Schwarzschild"* and *"Lieber*

Sommerfeld". In the coming years they would correspond about many other scientific issues and become friends [Sommerfeld 1916].

As a consultant for the geophysical applications, Klein and Sommerfeld turned to Emil Wiechert. Wiechert, like Sommerfeld, had begun his career in Königsberg. In 1897, Wiechert was called to Göttingen as professor of geophysics. "Klein told me that you will soon turn up [*kreiseln*] again in Göttingen for discussions about the top," Schwarzschild alluded to the collaboration between Sommerfeld and Wiechert in the spring of 1903, when the astronomical work was finished and the geophysical part of Volume III was in the making [Schwarzschild 1903]. Like Schwarzschild, Wiechert was an outstanding expert. His Göttingen institute became the nursery of a world-famous school of geophysics [Mulligan 2001]. Wiechert's expert knowledge entered the book not only in the form of consulting, but also as a part of the content. In 1897 he had published a shell model for the composition of the Earth. According to this model, the Earth's interior consists of an iron core that is surrounded by a crust of much smaller density. When Sommerfeld discussed the fourteen-month periodicity in the motion of the Earth's axis (the "Chandler wobble" discovered by the American astronomer Seth Carlo Chandler in 1891) as a peculiar top-phenomenon, he referred to Wiechert's shell model as the most recent explanation (cf. Vol. III, Ch. VIII, §7).

Thus the Göttingen advisors contributed to making the theory of the top relevant for contemporary astronomical and geophysical debates—a virtue that made this volume, as Sommerfeld wrote hopefully in his preface, "useful not only for the mathematician and physicist who studies mechanics for its own sake." It is ironic that the subjects which were at this time closer to Sommerfeld's work as professor of mechanics in Aachen, such as gyroscopes for torpedo guidance and ship stabilization, had to wait for Volume IV, which appeared only in 1910, when Sommerfeld had become professor of theoretical physics in Munich and no longer counted these subjects in his major area of interest.

Michael Eckert
Deutsches Museum, Munich

Advertisement for Volume III of the Theory of the Top.

After a long interval caused by my passage to a new teaching position, a third volume now follows the second (1898) volume of the *Theory of the Top*. This third volume is not, as it was intended to be, the conclusion of the work; it happened, namely, that the subject matter expanded enormously as soon as the general mathematical schema was applied, in accordance with the original plan of the work, to particular experimental conditions or to the many problems of the various special sciences with an interest in the theory of the top. As a result, this volume presents only applications of the theory to astronomy and geophysics; the technical and physical applications remain for a fourth (final) volume.

While the beginning of the present volume depends upon the content of the preceding chapters and treats, in an appendix to Chapter VI, of the top on the horizontal plane (through approximate calculations with rigorous error estimation),* the content of Chapter VII extends essentially beyond the circle of problems that are usually addressed in the analytic mechanics of ideal mechanisms. The general empirical facts concerning the effects of friction are presented, and, in association, friction at the support point of the top and its effect of uprighting the axis of the top are discussed in detail. Since, on the one hand, the experimental foundations of the theory of friction are not very certain, and, on the other hand, the mathematical difficulties of a rigorous development of the theory would be very great, the treatment is carried out in part graphically, with auxiliary assumptions, omissions, and approximation methods, as has been repeatedly recommended at earlier stages of the theory of the top. The precision of these methods suffices completely, in so far as one keeps in mind the appropriate goal: to create a clear qualitative image of actually observed phenomena, and to supply a quantitatively accurate description within the error bounds of the observations. In addition to friction at the support point, air resistance and elasticity of the top material and the support are considered as further causes of disfigurement for the ideal motion of the top. These investigations are partly important for later applications,

*The appendix to Chapter VI appears in our translation of Vol. II. — *Trans.*

and are partly intended to serve as examples of the mechanics of actual phenomena, or, as it is occasionally expressed here, terrestrial mechanics (in contrast to celestial mechanics, in which the influences treated here do not come into consideration, or to pure analytic mechanics, in which such influences are usually neglected in favor of an elegant mathematical development). In an appendix to this chapter, the treatment of the top on the horizontal plane is supplemented by the consideration of friction, with the enlistment of experimental data.

Chapter VIII treats in a first part of astronomical applications of the theory of the top, and in a second part of geophysical applications.

In the classical problems of the precession and the forced nutation caused by the motion of the Moon, new results can hardly be produced. The subject has been treated so exhaustively that the present exposition is aimed merely at providing the nonspecialist an intuitive procedure to replace the sometimes obscure manner of presentation of the astronomers. The means for this is afforded by a method of Gaufs for perturbation calculations, which is extended here in various directions.

Some of the problems investigated in the geophysical part, in contrast, are of the most recent date. We consider, in particular, the free nutation of the axis of the Earth, whose period was established by Chandler, and, furthermore, the phenomena of pole oscillations in general. In the presentation of the objective state of affairs and in the explanation of the same, the treatment given here may offer decided advances. Because of the fundamental importance of the problem, the auxiliary theorems from hydrodynamics and elasticity that are required for the explanation of the fourteen-month Chandler period are taken up and proven in the simplest manner. In addition, the theory of meteorological transport is developed for the explanation of the yearly period of the pole oscillations, where once again the previously emphasized impulse theory and the free and intuitive conception of the dynamic differential equations prove particularly fruitful. The conclusion of the geophysical part is formed by a discussion of the famous Foucault top experiment for the proof of the rotation of the Earth. Here we eliminate the unnecessary mathematical difficulties that occupy a large space in the older presentations of the Foucault experiment, and emphasize instead the perturbing influences and the estimation of their order of magnitude.

At the wish of my highly esteemed teacher F. K l e i n, I must point out, finally, that in the writing of this volume I have exceeded the content of the original university lecture given by Mr. Klein to a still greater degree than in the previous volumes. The mechanical analysis of the

perturbations in Chapter VII was only postulated in general outline in that lecture; the integration and approximation methods that are required here (and also in the appendix to Chapter VI), as well as the final results, are due to me alone. For what concerns the astronomical applications, Mr. Klein recognized the advantage of the Gaußian procedure, according to which he treated, in particular, the precession problem in his lecture; I myself, in contrast, have added the application of the same procedure to the problem of the nutation, as well as all numerical matters. The problem of the pole oscillations was not at all considered in Mr. Klein's lecture, so that the possible advances offered here (Chap. VIII, §6–8) are to be considered as my own. The foundation lines for the conception of the Foucault experiment were already drawn by Mr. Klein.

Moreover, I would like to emphasize that the constant interest that Mr. Klein has taken in the continuation of the work, as well as the encouragement that he has provided to me by many discussions and corrections, have essentially lightened my labor. Further, I must thank Mr. Schwarzschild and Mr. Wiechert for many valuable verifications and emendations in the astronomical and geophysical subjects.

May the present volume be of use not only to mathematicians and physicists who study mechanics for its own sake and will advance to a deeper and more lively understanding of the science by the availability of the thoroughly developed examples here, but rather may the representatives of astronomy, geophysics, and engineering also draw with pleasure upon this and the following volume as often as they come, in their particular fields, into contact with the theory of the top!

Aachen, July 1903. **A. Sommerfeld.**

Volume III

Perturbations. Astronomical and Geophysical Applications.

Chapter VII.

Theory and reality. The influence of friction, air resistance, and elasticity of the material and the support on the motion of the top.

§1. The contrast between rational and physical or celestial and terrestrial mechanics.

Abstract mechanics, which seeks to derive all phenomena of motion by mathematical deduction from a few fundamental principles, is traditionally designated as *rational* (= deductive) mechanics. The treatment of mechanical problems that accounts for reality in a more extensive manner by the use of empirical facts and experiments proceeds alongside, somewhat bashfully, as *physical* (= inductive) mechanics. In view of the vast differences between the results of the abstract theory and the facts of reality, however, one may raise the question whether, for the majority of applications, physical mechanics is actually rational, and the so-called rational mechanics is, in truth, most highly unphysical and irrational.

The differences between theory and reality that continually confront us are due, as everyone knows, to the occurrence of energy-absorbing or energy-dissipating forces. The abstract theory prefers to suppose that these forces have only a secondary significance, and may thus be treated as phenomena of the second order that can indeed blur but not completely disfigure the overall picture. A somewhat crassly chosen example may show us the extent to which this supposition is valid.

The German Model 88 infantry rifle[200] imparts to the bullet an initial velocity of approximately 620 $\frac{\text{m}}{\text{sec}}$. (The observation of this velocity is made about 25 m beyond the muzzle.) If we ignore air resistance and air friction, the energy-dissipating effects that come into play here, then the flight trajectory is the well-known parabola, and the greatest

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firing range is attained for an initial angle (elevation angle) of 45° with respect to the horizontal. If v is the common value of the initial horizontal and vertical velocity, H is the greatest elevation of the shot, and W is the range of the shot, then

$$v = \frac{620}{\sqrt{2}} \frac{\mathrm{m}}{\mathrm{sec}},$$

and one calculates, according to the elementary laws of projectile motion,

$$H = \frac{v^2}{2g} = \text{ca. } 10 \text{ km}, \quad W = \frac{2v^2}{g} = \text{ca. } 40 \text{ km}.$$

If, however, one opens the infantry firing regulations, which in this domain summarize a rich set of observational data, one finds on page 17 that the greatest observed firing range is approximately 4 km, and that it corresponds to an initial angle of 32°. The greatest elevation of the shot will be about ½ km for this trajectory, and lies not at the midpoint, as in the abstract projectile motion, but rather at a distance of 2,2 km from the muzzle.

The adjacent figure speaks even more clearly than the given numbers. It shows that the trajectory calculated under the neglect of air resistance ("calc.") gives not even a remote approximation to the observed trajectory ("obs.").

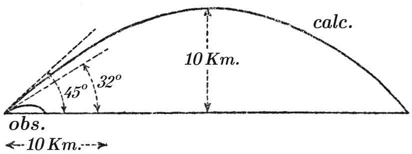

Fig. 71.

It must be admitted that no one would think of neglecting air resistance in ballistic problems. It can also be conceded that the influence of air resistance will not be as significant for smaller velocities as for the tremendous velocities of modern firearms. It is known theoretically, and confirmed by observation, that air resistance must increase rapidly with increasing velocity, particularly in the neighborhood of the speed of sound. Nevertheless, our example may be an appropriate warning against the overestimation of the results of rational mechanics, and the underestimation of so-called "secondary circumstances" such as friction, which can easily, on occasion, become the primary consideration.

The aversion of mathematical writers to friction problems is already present in the great founder of analytic mechanics, Lagrange.

Nowhere in his work does he mention the friction of one rigid body upon another. This unworthy example is also followed by K i r c h h o f f in his *Lectures on Mechanics.* Yet friction, after gravity, is the most important force in our existence. It is usually regarded as something detrimental and undesirable. In fact, the energy losses and thus the operational cost of all our machines, vehicles, etc., depend for the most part on some kind of friction. It would be unjust, however, to fail to recognize the beneficial side of friction. It is only friction that enables us to move forward at will on the Earth, be it through the force of our limbs or with the help of a conveyance. One may recall, for example, that a streetcar is unable to move if the beneficial effect of friction between the wheels and the rail is weakened by the formation of ice. It is friction, further, that enables me to hold the pen between my fingers, if only—according to the law of friction to be stated below—I press with sufficient force against the holder. Friction prevents the books that lie on my writing table (certainly not exactly horizontal) from sliding to the Earth as a result of gravity; it prevents the mountain that lies before my window from entering the valley and burying the city.

Why is it, we ask ourselves, that in spite of its decisive importance for all earthly phenomena, friction finds such slight regard in theoretical mechanics? One reason is of historical nature.

The oldest field of application of theoretical mechanics, and the oldest branch of mathematical science in general, is *astronomy.* The founders and primary developers of mechanics—G a l i l e i, N e w t o n, L a g r a n g e, L a p l a c e—had essentially astronomical questions in mind in their investigations. Thus it occurred that theoretical mechanics took on a garb that was essentially tailored to astronomical purposes. Now the heavenly bodies move in empty space without perceptible friction, and can, for most purposes, be regarded as single mass particles. Thus in celestial mechanics—but also only here—frictional phenomena withdraw. The problem of central interest is the problem of n mass particles that move freely in space while acting upon one another with conservative forces, a problem that constitutes the principal subject of

lectures and textbooks on theoretical mechanics, a problem which, however, is never realized in earthly occurrence.

The state of affairs is described more clearly if one speaks, rather than of rational and physical mechanics, of *celestial* and *terrestrial mechanics*. The form of theoretical mechanics that has been handed down to us has its origin and its appropriate domain of application in celestial mechanics. In order to be applied to the often obscure and complicated phenomena on Earth, it must be essentially supplemented by experimental material.

Another reason for the slight regard that theoreticians give to friction problems is the limited validity of the physical foundations of the theory of friction. We are aware from the outset that the usual laws on the sliding friction of solid bodies upon one another, on air resistance, on the internal friction of fluids or solid bodies, etc., are only rough approximations, and that the physical details of these processes are extremely complicated, and perhaps not at all able to be encompassed in formulas of general validity. A natural consideration of air resistance, for example, begins from the entrained motion of the surrounding air. The energy loss that the air resistance produces would be calculated, on the one hand, from the internal friction of the air, and, on the other hand, from the transfer of kinetic energy to the more remote layers of air. In very rapid motions, which are accompanied by a perceptible compression and rarefaction of the air, the energy transfer occurs at the speed of sound; thus in addition to the inertia of the air, elasticity and thermodynamics must also be taken into account. A glimpse into the multiplicity of the phenomena that are present here is granted to us by the beautiful and well-known instantaneous photograph of the air-waves generated by a bullet. Compared to this image, a formula that expresses the air resistance in terms of any power of the velocity must look very poor. It is just as hopeless, in view of the complicated play of waves in the vicinity of a steamship, to capture the ship's resistance in a simple formula. In any case, only extensive experiments carried out with the greatest means will provide any information here. Theory and calculation can, in this domain, sooner serve as guides to the judicious construction of experiments than for the prediction of the phenomena. In many cases, the theory has been able to provide rules (so-called similarity laws) for applying results from model experiments on a diminished

scale to the larger scales of actual problems. The achievements of the theory are very valuable in such cases, but are still much more limited than, for example, in problems of celestial mechanics.

The magnitude of the resistance in the preceding examples will also depend on the particular form of the bullet or the ship. It can very well be that a small change of form can effect a large change in the law of resistance. It is similar for sliding friction. Apparently minor circumstances are much more influential than one would expect and wish. A surface altered by abrasion behaves differently from a freshly worked surface. Small pieces of abraded material or specks of dust between the sliding surfaces can influence the magnitude of the friction considerably; moisture condensed from the surrounding air can act as a lubricant, and the law of friction can be changed not only quantitatively, but also in a fundamental qualitative manner. We see here the operation of a principle that is most highly inconvenient and disturbing for the natural scientist: small causes, large effects. A warning against the uncritical use of numerical results for friction must be made on these grounds. Numerical values that are found for certain experimental conditions need not hold for others. To give these values to two or three decimal places, as is often done in technical handbooks and in many textbooks on experimental physics, has, in any case, no value.

One thus understands that the laboratory physicist who is in the pleasant position to choose his problems freely, and in part according to the aesthetic point of view, will prefer to pass over friction problems, since he can promise himself no pure and general laws from his studies. The technical worker stands in a different relation to the laws of friction, which for him are matters of vital importance. Thus the more recent contributions to knowledge of the laws of friction arise essentially from the technical side, as we will discuss in the following section.

It further follows from this state of affairs, however, that the mathematical treatment of friction problems must be carried out from a point of view different from that of the problems of rational mechanics. The mathematician seeks, according to his education and custom, to solve the problems before him with complete rigor, so that a calculation to arbitrarily many decimal places is theoretically possible. Considering the great accuracy of astronomical observations, this method is in fact

appropriate for problems of celestial mechanics; for all problems, however, in which frictional influences are essential—that is, all problems of terrestrial mechanics—such a precise calculation would be unattractively incompatible with the precision of the physical foundation. Here it is indicated to seek not a quantitative calculation, but rather a qualitative understanding of the phenomena, and, when one does proceed quantitatively, to calculate from the beginning not with an arbitrary but with a bounded precision. The differential equations in the theory of the top that include friction terms, for example, become quite complicated, and, as one usually says, "may not be integrated." To the contrary, we emphasize the basic principle that one *should* not integrate such equations; one should interpret them and construct their solutions, as we will seek to do in the following.

The preceding discussion gives the motivation and the sense of our detailed treatment of friction problems for the top. Although we go beyond what has previously been done, we fall short of the desired goal; the lack of an adequate experimental foundation must restrict us to a schematic treatment of the problem.

§2. Report on the laws of friction.

Our knowledge of the laws of friction was founded, as is well known, by C o u l o m b.[201] We enter into these laws in some detail, since the relevant questions are generally unfamiliar in theoretical circles.*)

Coulomb found that a force appears at the boundary of two rigid bodies that slide upon one another, a force that opposes the motion of each body with respect to the other, and that is proportional to the total normal force with which the two bodies are pressed together. The factor of proportionality is called the *coefficient of friction* (or, more precisely, the coefficient of kinetic friction). This coefficient is to be regarded as a material constant, or, more correctly, as a characteristic

*) For further orientation, we refer to the very comprehensive treatment of friction problems in the textbook by J. P e r r y, Applied Mechanics, New York 1898, which we have used repeatedly for this and the previous sections.[202] An informative report on the entire literature of friction is due to F. M a s i: Le nuove vedute nelle ricerche theoriche ed experimentali sull' attrito. Bologna, Zanichelli 1897.[203]

constant for the material and the surface condition of the two bodies. The friction force and the coefficient of friction should thus be independent of the sliding velocity and the size of the contact area, or, for equal total normal force, independent of the magnitude of the specific normal force, the normal force per unit area of the contact surface. The formula for the Coulomb friction law is, if one denotes the coefficient of friction by μ, the total normal force by N, and the magnitude of the frictional resistance by W,

$$W = \mu N.$$

We first wish to restrict this statement to *dry friction*; it is invalid for friction in the presence of a lubricant. Further, we must recall the well-known difference between kinetic (dynamic) friction and static friction. We explain static friction in the following manner.

If the applied force that acts to produce the motion of a test body with respect to its support, or the relative motion of the two, is not sufficient to overcome the friction, we will ascribe to the friction only the magnitude of the applied force that holds the body in equilibrium. This remains valid until the applied force exceeds a limiting value at which motion occurs, a value that is again proportional to the normal force N. The factor of proportionality may be denoted by μ_0, and is called the *coefficient of static friction.* The law of static friction can thus be expressed by the equation

$$W \leq \mu_0 N.$$

The coefficient of static friction is generally substantially larger than that of kinetic friction. This circumstance, as well as the law of friction in general, is illustrated in a beautiful experiment given by G. Herrmann,*) an experiment that anyone can repeat without any difficulty.

A stick is placed horizontally on the two index fingers. The fingers are then brought together. At which finger does the stick begin to slide? At the finger that stands farther from the center of gravity of the stick; for the normal forces N_1 and N_2 that the stick applies to the two fingers are, according to the law of the lever, proportional for each finger to

*) Der Reibungswinkel, Festschrift zum Jubiläum der Univ. Würzburg, 1882.[204]

the distance from the center of gravity to the other finger. The normal force is therefore smaller for the finger at the greater distance. According to the law of friction, the frictional force that opposes the sliding here is also smaller than for the other finger; the sliding must therefore begin here.

The stick now slides on this finger, and not only until the distance of this finger from the center of gravity is equal to that of the other, but rather somewhat farther, because $\mu_0 > \mu$. When the distances of the fingers from the center of gravity are in the proportion $\mu : \mu_0$, a change in the motion occurs; the stick now rests on the finger on which it previously slid, and begins to slide on the other. For the normal forces N on the two fingers are inversely proportional to the distances to the center of gravity, and are therefore, at this instant, in the ratio $\mu_0 : \mu$. The kinetic friction at one finger will then equal the static friction at the other, and would, if the sliding continued in the previous sense, become even greater, which is obviously absurd. The sequence of events is continuously repeated, so that the distance from the center of gravity to the finger on which the stick slides diminishes each time to the $\left(\frac{\mu}{\mu_0}\right)^{\text{th}}$ part of the distance from the center of gravity to the other finger. As a result, the distances from the center of gravity to the two fingers will be diminished alternately, and, when the fingers have come together, the stick will be supported at its center of gravity, and therefore freely oscillate!

In addition to the intuitive illustration of the friction law, the experiment also permits of a measurement of the ratio $\mu_0 : \mu$. It is enough, for this purpose, to mark a few reversal points of the motion of the stick, and to measure their distances from the center of gravity. The ratio of the distances between two successive reversal points provides the desired ratio of the coefficients of friction. The totality of the reversal points forms, on both sides of the stick, the successive points of a geometric series. If the stick is not too short, the measurement can be relatively precise.

The difference in the values of μ_0 and μ suggests the hypothesis of a continuous passage between the value μ_0 corresponding to the sliding velocity zero and the value μ corresponding to perceptibly larger sliding velocities. This conjecture has been confirmed by the experiments of J e n k i n and E w i n g.*) For materials with significantly different

*) London Philos. Transactions, Vol. 167 (1877), p. 509.[205]

coefficients of friction μ_0 and μ, the continuous passage can in fact be detected.

If we therefore wish to claim, with Coulomb, the independence of the coefficient of friction on the velocity, we must first exclude the domain of very small velocities. How well is this independence now confirmed for greater velocities?

The older experiments of General Morin, which were performed in the years 1831–1833 in Metz and encompassed a velocity range up to $4\,\frac{\mathrm{m}}{\mathrm{sec}}$, appear to confirm the independence. To this day, Morin's experimental results form the permanent supply of numerical specifications for coefficients of friction in technical handbooks and textbooks on experimental physics. A high degree of reliability, however, can hardly be ascribed to them, since, according to what was said in the previous section, such numerical results depend strongly on the secondary circumstances of the experiments.[206]

The behavior of the coefficient of friction for higher velocities remains, in any case, an open question that first became relevant in the development of railroads. Between the brake pad and the collar of the wheel we have true sliding friction without lubrication, while the friction between the wheel and the rail is generally, at least by intention, rolling friction, and becomes sliding friction only in the exceptional case that the wheels skid on the rails. Concerning the friction between the brake pad and the wheel, the experiments of *Douglas Galton*,*) which were carried out in full scale on various English rail lines with the support of the Westinghouse firm, stand out above all others. Galton's experimental van contained an entire series of tachometers and dynamometers. The tachometers provided the rotational velocity of the wheels and the progressional velocity of the van. The sliding velocity between the wheel and the brake pad is equal to the first of these velocities, and the possible sliding velocity between the wheel and the rail is equal to the difference of the two. The dynamometers measured 1) the braking force with which the brake pad was pressed to the wheel, and therefore the force that acts as the normal force N for the friction between the brake and the wheel; 2) the frictional resistance W between the brake and

*) Institution of Mechanical Engineers Proceedings 1878, 1879; see, in particular, 1879, p. 172 or Engineering 1879, p. 371 or Reports of the British Association (Dublin) 1878.

the wheel; 3) the resistance against the forward motion of the car per axle, which comprises air resistance, rolling friction on the rail, etc., which, however, if sliding on the rails occurs, represents in essence the frictional resistance W' of this sliding. All these tachometers and dynamometers registered automatically, and thus provided the temporal changes of the relevant velocities and forces. The very interesting diagrams that were obtained for these quantities can only be indicated here. We must restrict ourselves to the conclusions to be extracted from them regarding the variability of the friction coefficients. The coefficient of friction between the brake pad and the wheel results from division of the measured forces W and N, while the coefficient of friction between the wheel and the rail follows from the likewise measured friction W' and the portion of the van weight G that falls on the individual wheel.[207]

The corresponding friction coefficients for different experiments with the same velocities naturally show no complete agreement among themselves; their magnitudes depend on the weather and the related dampness condition of the sliding surfaces, as well as on the cleanliness of the surfaces and the braking duration, which obviously causes an increase in temperature and thus a change in the surface condition. It occurred, for example, that after the braking force had acted for 20 seconds, the friction coefficient was reduced to half of its original value. Nevertheless, the mean of the friction coefficients obtained from many (up to 100) experiments show a clear regularity; namely, a *continuous decrease of the friction coefficient with increasing velocity.* Fig. 72 gives the coefficient of friction between the brake pad and the wheel (brake pad of cast iron, wheel rim of steel). The solid line is the mean value of the observations, and the dotted lines are the largest and smallest values for each velocity. The entire strip between these boundary lines is to be imagined

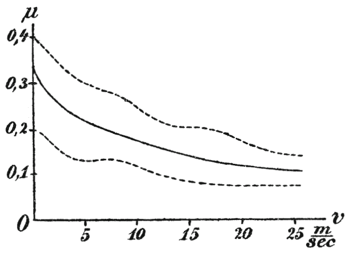

Fig. 72.

as filled with observation points that thicken along the mean line. One first recognizes the difference between the static coefficient of friction and that of the considered motion. But the figure further shows that the

coefficient of friction for the velocity 60 km/hr (= 16,7 m/sec = mean express train velocity) amounts to less than half of the static coefficient of friction (0,33), and that the coefficient of friction for 90 km/hr (= 25 m/sec = highest current permissible speed in Germany) amounts to no more than a third of the static coefficient of friction. This naturally leaves the dependence between μ and v uncertain, and thus expressible through one formula or another. We thus disregard the statement of such a formula as well as the numerical values, since the figure depicts all that is to be concluded from the degree of precision of the observations, and since any formula used to represent the observations would be highly arbitrary. — Concerning the coefficient of sliding friction between the rail and the wheel, Galton's experiments give a less certain conclusion; as much as can be discerned from them without doubt is that this coefficient also decreases continually with increasing velocity from its greatest value for small velocity.

Older French and more recent experiments conducted in Germany*) yielded essentially the same results.

But it is to be considered that Galton's experiments refer to a very extended velocity range. For a moderately changing velocity, the variation of the friction coefficient is also only small; according to the curve one has, for example, the Galton mean value of approximately $\mu = 0{,}27$ to 0,23 for $v = 2$ to 6 m/sec. These differences are, with respect to the general uncertainty of friction values, undoubtedly to be neglected. Coulomb's assumption of a velocity-independent coefficient of friction is thus affirmed, in the first approximation, for a *moderate* velocity range. For our application to the top, in which velocities of the order of express train velocities certainly do not come into consideration, we may thus set, with Coulomb, the coefficient of friction equal to a constant.

The lack of dependence of the coefficient of friction on the size of the contact area that is stated in the Coulomb law may also be tested experimentally. According to the Coulomb law, a prism of base area 10 cm^2 and height 4 cm that moves on a fixed plane must be subjected to the same frictional force as a prism of base area 20 cm^2 and height 2 cm, since, assuming the same material, the total normal force against the support, namely the weight of the prism, is the same in each case,

*) Cf. Organ der Fortschr. des Eisenbahnwesens 1889, p. 114. The brake pad consisted here of cast steel. The experiments were conducted not on the track, but rather in a workshop.[208]

while the normal forces per unit area in the two cases have the ratio $2:1$. It appears that this result is well confirmed in experiments,*) in so far as a perceptible deformation of the support is not caused by a very large specific pressure. —

It is also well known that the process of rolling two surfaces upon one another, which (apparently) occurs without sliding, is also associated in a small degree with energy dissipation. The law of *rolling friction* that is usually adopted for the calculation of this energy loss was likewise constructed by *Coulomb*. Here it is useful to speak not of a *frictional single-force*, as for sliding, but rather of a frictional moment that is to be overcome at each instant by the turning-moment that is employed for the rolling. If the total normal force at the contact between the support and the "roller" is again denoted by N and the frictional moment is denoted by M, then one sets

$$M = \nu N.$$

The quantity ν is called the *coefficient of rolling friction*; as the equation shows, it is not a pure number like the coefficient of sliding friction, but rather has the dimension of length. This coefficient is also to be regarded as a material or surface constant. If one wishes to replace M by a frictional single-force W on the outer surface, then one must set the latter equal to M/r, where r is the radius of the roller, or, for a non-circular perimeter of the roller, is equal to the radius of curvature at the considered place on the circumference; this single-force W is therefore directly proportional to the normal force and inversely proportional to the radius.

A beautiful work of O. Reynolds**) gives some information on the mechanism of rolling friction. Reynolds shows that rolling is always accompanied by a certain sliding due to elastic deformation.

If one assumes, for simplicity, that the support is essentially softer than the roller (support of rubber, roller of iron), then one can disregard the deformation of the roller, and consider only that of the support. The latter will consist of a trough-shaped depression, so that the support is stretched in the middle of the depression, and is raised and

*) Perry l. c. p. 67.

**) On rolling Friction, London Philos. Transactions Vol. 166, Part I (1876) and Ges. Werke Bd. I, p. 110.

compressed on the sides. The contact no longer occurs at a geometric point, but along the surface of the trough, whose midpoint we denote as the mean contact point P. In the adjacent figure, a number of points are marked on the roller and the support. The points on the roller are equidistant. The points on the support were equidistant before the deformation; the changed distances thus show the sense of the distortion that has occurred. The points marked by the same numbers are the mean contact points that successively coincide with one another in the rolling process, which is made possible by the elongation of the support at the mean contact position due to the rolling process. In an instantaneous state, however, these points do not coincide. Point 4 of the instantaneous state must, for example, slide along the small section of the rolling circumference between the two points 4 in the figure before it has become the mean contact point. The same holds for each point of the contact surface. In this process, therefore, a certain sliding friction in fact occurs.

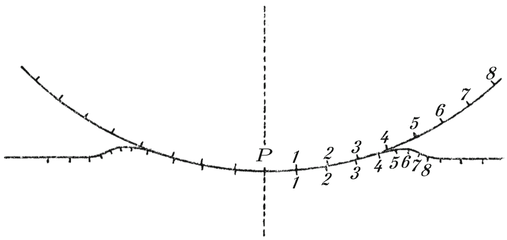

Fig. 73.

The total energy loss of the rolling friction will be made up in part by the energy loss associated with the sliding friction, and in part by the work required in the elastic deformation, in so far, namely, as the latter occurs irreversibly.

Reynolds was able to demonstrate experimentally that the representation in Fig. 73 is correct. Under the conditions assumed above, the circumference of the roller develops not on the natural upper surface of the support, but rather on the extended surface at P. If the path that the roller has covered after one rotation is measured on the support that has returned to its natural length, then this path must be somewhat shorter than the circumference of the wheel. Reynolds has, in fact, demonstrated this experimentally for the case in which the roller is harder than the support. The opposite must occur, and according to Reynolds does occur, if the support is considerably harder than the

roller. If both are of the same material, then the path measured on the support is again somewhat smaller than the circumference of the roller, as would be shown by a more detailed entrance into the preceding deliberation.

The Reynolds investigation is not carried out to a precise measurement of the magnitude of the frictional resistance and a verification of the Coulomb assumption. It is not very probable that this simple assumption corresponds precisely to reality, considering the complicated nature of the process. —

In addition to sliding and rolling friction, one speaks in the third place of *boring* friction, particularly when one body rotates on another about the normal to the contact point. Since the contact is assumed in this case to be pointlike, and since the contact point is assumed to be a point on the rotation axis, sliding of the two bodies on one another does not, theoretically, take place. This circumstance has given occasion for the introduction of the special term "boring friction." However, the process is reduced immediately to sliding friction if only one assumes a somewhat extended contact between the bodies. One can then speak of a mean radius a of the contact surface, and may reduce the forces distributed on rings about the rotation axis to sliding friction at this mean distance a. These forces may obviously be composed into a turning-moment about the normal to the contact surface, whose magnitude is calculated from the law of sliding friction as

$$M = \mu' N, \quad \mu' = \mu a.$$

The factor of proportionality μ' can be denoted as the *coefficient of boring friction*; it has the dimension of length and depends, in addition to the material and the nature of the surface, on the extent of the contact surface. It should naturally not be claimed from the equation $\mu' = \mu a$ that the coefficient of boring friction would be predetermined from that of sliding friction if one could measure the size of the contact surface. Rather, this equation provides only an indication of the meaning of the coefficient μ', and an approximate magnitude of the ratio of boring to sliding friction that will be of use to us in the next section.

We have explicitly restricted ourselves in this report to *dry* friction, even though friction under the application of a *lubricant* is of predominant interest in practice. The current understanding is that lubrication friction is subject to a completely different law; it depends, namely, on

the law of internal friction for a viscous flow, while the older technical literature treats it according to the Coulomb schema of dry friction. We can now (with Petroff and Reynolds) speak of a *hydrodynamic theory of lubrication*, a theory that is confirmed, to the extent that one can expect considering the difficulty of the subject, by rationally constructed experiments.*) The actual physical understanding of lubrication friction has been essentially advanced in this way. Will it be possible, we wish to ask in conclusion, to similarly advance our physical understanding of dry friction, in that we conceive the remaining air film between the frictional surfaces as a kind of lubricant, and apply to it the friction laws of kinetic gas theory? And, further, to what extent is friction always associated with abrasion of the surface?

§3. Qualitative considerations on sliding and boring friction for the top.

In order to study the influence of friction on the top with a fixed support point, we must first examine the circumstances under which friction occurs here. Our conception of these circumstances is certainly very schematic and idealized, and may, from case to case, depart significantly from reality, depending on the particular properties of each individual device.

We consider, for example, the two instruments that are illustrated on pages 1 and 2. If we may completely disregard friction, then the motion of the two instruments follows according to precisely the same laws, in so far as the masses of the rings on page 2 are neglected with respect to the mass of the inner rotor. With consideration of friction, however, the two instruments behave completely differently.

In the instrument of page 2, frictional forces appear at the three bearings that carry the axes of the outer ring, the inner ring, and the rotor. If the bearings are sufficiently lubricated, we have here a case of fluid friction in the sense of the previous section. In any case, the effect of friction will consist of a turning-moment that opposes the instantaneous rotation of each of the three axes. Since the motion about each of the three

*) Cf. here the report of Masi cited above. We refer further to the comprehensive and technically distinguished experiments of R. Stribeck: Die wesentlichen Eigenschaften der Gleit- und Rollenlager. Ztschr. des Vereins deutscher Ingenieure 1902, Nos. 36, 38, and 39.[209]

axes is given directly in terms of the Euler angles ψ, ϑ, φ, an additional term that signifies the frictional moment about the relevant axis would appear in the Lagrange equation for each of these angles.

We do not intend to enter into this further, but rather restrict ourselves to the model of page 1. Here we must first consider the form of the lower end of the figure axis, and the form of the seat that bears it. We wish to assume that the lower end of the figure axis is *spherical*, and that the seat is a *circular cone*; the surfaces are assumed to be dry and not elastically compliant. The contact between the sphere and the

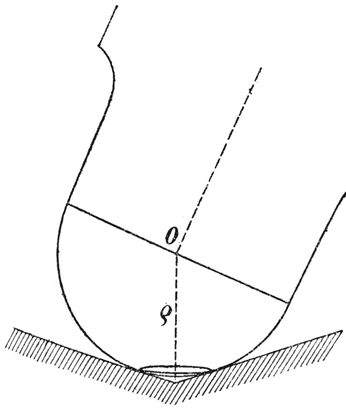

Fig. 74 a.

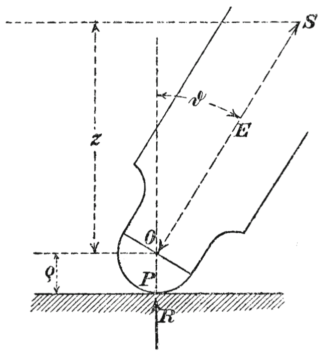

Fig. 74 b.

cone then always occurs on a fixed horizontal circle. The sphere is displaced into itself for all motions of the figure axis. Its midpoint therefore remains exactly fixed in space. *In this midpoint we have the assumed stationary point O of the top.*

In the model of page 1, the cone that bounds the seat is very flat; for a theoretical calculation of the frictional forces it is convenient, or even necessary, to assume that the seat is absolutely flat, and therefore to let the cone degenerate into a plane. The small circle in which the sphere would touch the flat cone then contracts into a *point* that always lies directly below the midpoint of the sphere. The concept of the "support point" is thus decomposed into two concepts: *the fixed point O = the midpoint of the sphere*, and *the contact point P = the limit of the just-named small contact circle.*

We would certainly not hide the fact that we remove ourselves decisively from the actual conditions of our problem in this manner, and that we no longer have, *after* passing to the limit, a top with a fixed

point, but rather, strictly speaking, a top that may move freely on the support. The fixity of the point O is lost as we let the cone become flat. It occurs here, as so often in applications, that a limit process is mathematically convenient but physically meaningless (one thinks of the limit process from molecular to infinitesimal dimensions in all of theoretical physics), and that we may go, according to the conditions of the problem, only *to a neighborhood of the limit*, and not *to the limit itself*. In all such cases, the tacit assumption is made that the mathematical treatment of the limiting case does not deviate essentially from the actual case, an assumption that will be confirmed, as a rule, by the results of the treatment. We wish to emphasize the corresponding assumption here explicitly: we assume that the lateral motion of the point O is excluded by an appropriate mechanical fixture on the support, but we may subsequently calculate the effect of friction without significant error as if the support were planar.

The passage from the original tangent *circle* to the present tangent *point* is necessary because we would otherwise fall into endless complications regarding the elastic deformation at the contact location. If we would operate with the contact *circle*, namely, then we must, in order to determine the frictional forces, first establish how the reaction of the seat on the top is distributed over the circumference of the contact circle. This is, however, one of the many important questions that remain undetermined from the standpoint of rigid-body mechanics; its answer must be drawn from the theory of elasticity. It is generally well known, where the seating of a rigid body is concerned, that only six unknown reactions are determined by the six equilibrium conditions of the usual statics in space. If more are present, the additional reactions remain *statically indeterminate*. For our contact circle, however, we have infinitely many reactions, since the bearing pressure at each element of our contact circle is unknown in magnitude and direction. The problem therefore belongs to the domain of elasticity. If we must, however, start to include elastic deformation in the calculation, then we must also consider that the contact *circle* is transformed, in fact, into a contact *area* because of the elastic flattening of the surfaces. The size of this area and the change in form of our spherical and conical surfaces must be determined by the theory of elasticity. Only if this is done could we give the distribution of the reaction pressure and the

magnitude of the friction. We circumvent the extraordinary difficulty that would result here by our assumption of an absolutely flat seat and a point contact.

Under this assumption, the counterforce, or the vertical (that is, the direction normal to the seat) reaction R of the seat, follows from the simple consideration presented on page 515 for the top on the horizontal plane. From the impulse theorem, namely, there follows, here as there,

$$R = M(g + z''), \tag{1}$$

understanding by z the vertical coordinate of the center of mass in the fixed xyz-coordinate system at the support point O (cf. Fig. 74b).

With respect to the sign of R, the following is to be noted. Due to its rigidity, the seat can produce, if necessary, an extraordinarily large *positive* reaction (by positive is understood the direction from below to above), but not the smallest *negative* reaction. As soon as such a force is calculated from (1) in the course of a specified motion, the seat would not suffice to ensure the equilibrium of the point O: the top would leave the seat for vanishing R, and its lower end would shoot upward. From then on it would no longer move as a top with a fixed support point, but rather would describe a Poinsot motion in free space about its center of gravity, while the center of gravity itself would move according to the law of falling bodies. We wish to exclude such motions from consideration here, and therefore assume that

$$g + z'' > 0$$

always holds. It is consistent with this assumption if we later assume that the center of gravity acceleration z'' is always very small compared to the gravitational acceleration g, so that we reduce the reaction force R to its "static" component

$$R = Mg = \text{the weight of the top}, \tag{2}$$

and disregard the "dynamic component" Mz''. This is one omission (**Omission I**) that we make in the interest of the feasibility of the friction problem; the validity of our results is thus restricted to a class of motions that we can designate as "precession-like." (For regular precession, it is indeed true that $z = \text{const.}$, and therefore $z'' = 0$; a motion can thus be called precession-like if z'' never assumes values very different from zero.)

The friction at the contact point P is known as soon as R is. We first distinguish sliding and boring friction, but note that a separate calculation of the two is doubtful, as will be discussed in §6. This doubt does not come into consideration in the following, since, as we will see, boring friction may be neglected with respect to sliding friction for a reasonably large inclination of the figure axis.

The sliding friction is a single-force applied at the point P, with magnitude

$$W = \mu R. \tag{3}$$

Its direction is horizontal, and it stands, just like the direction of motion of P, perpendicular to the instantaneous rotation axis OR (cf. Fig. 75). With respect to the reference point O, there thus arises a turning-force of magnitude

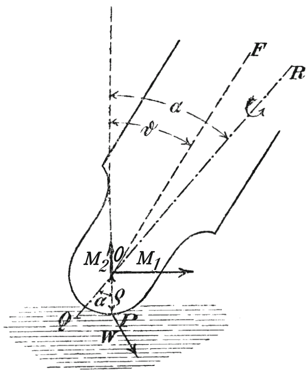

Fig. 75.

$$M_1 = \varrho\mu R, \tag{4}$$

where ϱ signifies the lever arm of W with respect to O; that is, the radius OP of the bounding sphere. The axis of this turning-force coincides with the direction of the horizontal component of the rotation vector.[210]

We calculate the boring friction by its moment M_2, which has the vertical OP as its axis and is opposed in sense to the vertical component of the rotation vector. Its magnitude is (cf. the previous section)

$$M_2 = \mu' R = \mu a R. \tag{5}$$

We wish to form a judgment of when one or the other frictional effect will dominate. For this purpose, we calculate the corresponding work losses $d\mathfrak{A}_1$ and $d\mathfrak{A}_2$ during a time interval dt. If Ω signifies the magnitude of the instantaneous rotational velocity, α the angle between the vertical and the rotation vector OR, and therefore $\Omega \sin\alpha$ the horizontal component and $\Omega\cos\alpha$ the vertical component of the rotation vector, then

$$d\mathfrak{A}_1 = -M_1\Omega\sin\alpha\, dt, \quad d\mathfrak{A}_2 = -M_2\Omega\cos\alpha\, dt, \tag{6}$$

and

$$d\mathfrak{A}_1 : d\mathfrak{A}_2 = \varrho\sin\alpha : a\cos\alpha = \varrho\,\mathrm{tg}\,\alpha : a.$$

We can speak of the quantity a, which in the previous section signified the mean radius of the contact surface, as the radius of the original contact circle from which our contact point P was formed. The quantity $\varrho \operatorname{tg} \alpha$, on the other hand, signifies (cf. Fig. 75) the distance of the contact point P from the intersection point Q of the instantaneous rotation axis with the horizontal plane. Our preceding proportion thus states that the work of the sliding friction is smaller or larger than that of the boring friction according to whether the instantaneous rotation axis passes through the contact circle or not. If we let the contact circle contract approximately to a point, it follows that boring friction is important compared to sliding friction only for a nearly vertical position of the rotation axis, and that, on the other hand, sliding friction absorbs considerably more work and thus has a significantly greater effect on the course of the motion for a perceptibly nonvertical rotation axis. We are thus entitled in the following *to generally neglect boring friction with respect to sliding friction* **(Omission II)**. Since the figure axis approximately follows the rotation axis for the motions to be considered, the named omission will be permissible as long as *the figure axis is not perceptibly vertical.*

We wish to express the work of sliding friction in a second manner; namely, in terms of the Euler angles φ, ψ, ϑ. We resolve, for this purpose, the rotation vector Ω into its three components φ', ψ', ϑ' with respect to the figure axis, the vertical, and the line of nodes. If we then project the path formed by the three line segments φ', ψ' ϑ' onto the horizontal plane through O, we obtain the horizontal component of the rotation vector. This becomes, according to Fig. 76,

$$\Omega \sin \alpha = \sqrt{\vartheta'^2 + \varphi'^2 \sin^2 \vartheta}.$$

There follows, according to (4) and (6),

$$(7) \qquad d\mathfrak{A}_1 = -\varrho \mu R \sqrt{\vartheta'^2 + \varphi'^2 \sin^2 \vartheta}\; dt.$$

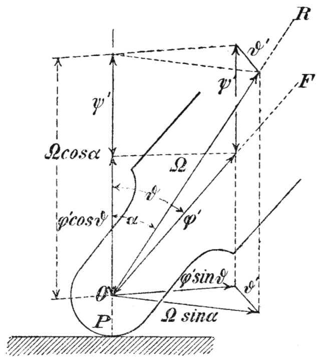

Fig. 76.

Through a small formal alteration, we can write this expression as a linear function of the coordinate changes $d\vartheta$, $d\varphi$, $d\psi$, as will be needed for the starting point of the Lagrange equations. We thus rewrite (7) as

$$(8)\qquad d\mathfrak{A}_1 = -\varrho\mu R\Big(\frac{\vartheta'\,d\vartheta}{\sqrt{\vartheta'^2+\varphi'^2\sin^2\vartheta}} + \frac{\varphi'\sin^2\vartheta\,d\varphi}{\sqrt{\vartheta'^2+\varphi'^2\sin^2\vartheta}}\Big).$$

The coefficients of $d\vartheta$, $d\varphi$, $d\psi$ in this expression (cf., for example, page 78) are called the "components of the sliding friction with respect to the coordinates ϑ, φ, ψ"; we write

$$(9)\quad \Theta_1 = -\varrho\mu R\frac{\vartheta'}{\sqrt{\vartheta'^2+\varphi'^2\sin^2\vartheta}},\quad \Phi_1 = -\varrho\mu R\frac{\varphi'\sin^2\vartheta}{\sqrt{\vartheta'^2+\varphi'^2\sin^2\vartheta}},\quad \Psi_1 = 0.$$

With respect to the magnitude of these friction components, we will likewise be guilty of an imprecision **(Omission III)**. For the most important motions of the top, the rotation vector always coincides approximately with the figure axis. The component φ' of the rotation vector with respect to the figure axis will therefore always be considerably larger than the component ϑ' with respect to the line of nodes. For regular precession, ϑ' will even be, disregarding friction, exactly zero. In that we stipulate that ϑ' *will be struck with respect to* φ' *in the expressions for the frictional work and the frictional forces*, we once again restrict our consideration to "precession-like motions."

In this sense, we write (7) and (9) as

$$(10)\qquad \begin{cases} d\mathfrak{A}_1 = \mp\varrho\mu R\varphi'\sin\vartheta\,dt, \\ \Phi_1 = \mp\varrho\mu R\sin\vartheta, \quad \Psi_1 = \Theta_1 = 0. \end{cases}$$

An explanation is wanted for the double signs in (10). It is clear that the square root in equation (7) is always to be calculated with the positive sign, since the frictional work is always negative. The same holds for the square roots in equations (8) and (9). If we expand these roots in ϑ'/φ', then we must set them equal, in the first approximation, to $|\varphi'\sin\vartheta|$; that is, equal to $\pm\varphi'\sin\vartheta$, according to whether φ' itself is positive or negative. This also holds, in particular, for the value of Φ_1, in which we have canceled the denominator $|\varphi'\sin\vartheta|$ against the factor $\varphi'\sin^2\vartheta$ of the numerator. The upper sign in equations (10) is therefore to be chosen in the cases where the top rotates about the figure axis in the clockwise sense ($\varphi' > 0$, Fig. 77a), and the lower sign in the opposite cases ($\varphi' < 0$, Fig. 77b).

We have now made all the preparations for the approximate solution of the friction problem that will occupy us in the next section. We will, in particular, have to account for the well-known observational fact that the top is, in the mean, slowly uprighted by sliding friction.

It is well, however, to make this fact plausible from the outset in a way that allows us to follow the effect of friction intuitively, even if in a very imprecise way.

We assume that the top is in rapid rotation and that the rotation axis is approximately coincident with the figure axis. The same then holds for the impulse axis, whose position is indeed determined from the positions of the figure axis and the rotation axis. For the sensible representation of the impulse vector, we must therefore lay off a very long vector OJ from O, and indeed in the approximate direction of the positive figure axis (upward) or in the approximate direction of the negative figure axis (downward), according to whether the top rotates in the clockwise or counterclockwise sense about the positive figure axis. The rotation itself will be represented by a vector OR, which is approximately directed just like the impulse axis, and therefore upward in one case and downward in the other. The first case is represented in Fig. 77a, and the second in Fig. 77b. The effect of sliding friction on the motion of the top is expressed, as we saw, in the appearance of a moment M_1 that has the same axis as the horizontal component of the rotation vector and the opposite sense. If we insert the horizontal component OH of the rotation vector into our two figures, the sense of the frictional moment is thus determined. The arrow that represents M_1 must run in Fig. 77a from right to left, and in Fig. 77b from left to right.

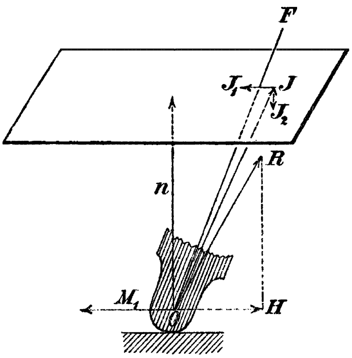

Fig. 77 a.

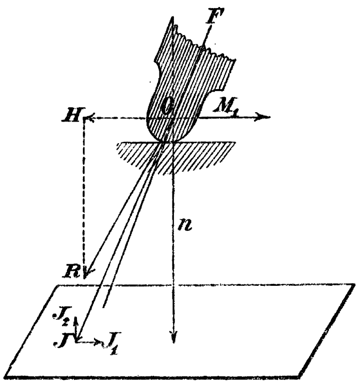

Fig. 77 b.

According to the fundamental property of the impulse vector, it is composed at each time increment dt with the additional impulse of the exterior forces. In so far as the latter is generated by the sliding friction, it is equal to $M_1\,dt$; the result of its composition with the impulse

vector OJ is indicated in the two figures. In both cases, the effect of the additional impulse is that the horizontal component of the total impulse is somewhat diminished, and that the vertical component is unchanged. *The impulse will thus be somewhat weakened in magnitude, and its direction will become somewhat more vertical. Its endpoint moves from* J *to* J_1 *in the horizontal plane through the endpoint of the initial impulse.*

The effect of friction on the direction change of the impulse will obviously be smaller as the current length of the impulse vector is larger; the addition of the small segment JJ_1, which depends only on the magnitude of the reaction R, the coefficient of friction μ, the sphere radius ϱ, and the time increment dt, affects a longer vector OJ less than a shorter. *The change of position of the initial impulse therefore follows more slowly as the initial impulse is stronger or the top initially rotates more rapidly.*

The impulse vector OJ will naturally be changed at the same time by the effect of gravity. On this basis, the endpoint of the impulse is displaced at each instant in the direction of the axis of the gravitational moment; that is, the direction of the line of nodes. Since, however, the line of nodes stands exactly perpendicular to the figure axis and approximately perpendicular to the impulse axis, as long as our assumption of the approximate coincidence of the figure axis and the impulse axis applies, the effect of gravity produces approximately no change in the magnitude of the impulse vector and its inclination with respect to the vertical. Our above claim with respect to the magnitude and change of position of the impulse therefore remains valid when gravity is considered. One notes, in particular, that the sense of the gravity moment, which depends on the position of the center of gravity on the figure axis, is irrelevant for our deliberation. *Thus the impulse vector will always be uprighted more and more by friction, whether the center of gravity lies above or below the support point.*

We wish now to show, however—and with this proof we first reach our actual goal—that the figure axis of the top behaves just like the impulse axis.

To this end, we first remark that the *rotation axis* follows the position of the impulse axis exactly for the spherical top, and approximately for the symmetric top. The figure axis, on the other hand, rotates

about the instantaneous rotation axis on a cone. And indeed this rotation is much faster, for a sufficiently strong eigenimpulse, than the change of the rotation axis itself, so that the rotation axis is displaced only slightly during one full revolution of the figure axis. In fact, the motion of the impulse axis, and thus also the motion of the rotation axis, will be slower as the original impulse imparted to the top is larger, while the rotation of the figure axis will be faster as this impulse is larger. Above a certain magnitude of the impulse, the motion of the rotation axis can therefore be considered as infinitely slow compared to the motion of the figure axis. The mean position of the figure axis then always coincides with the location of the rotation axis, and what was already established for the position of the rotation axis and the impulse axis also holds for the figure axis: *in the mean, the figure axis must also be uprighted by the effect of friction, and indeed ever more slowly as the initial rotation is faster.* This mean motion will be overlaid with a small oscillation or nutation of the figure axis that is caused by the progressive turning about the rotation axis, and that brings the figure axis alternately nearer and farther from the vertical. —

Perhaps it can be astonishing, at first glance, that friction has the consequence of uprighting the figure axis even when the center of gravity lies above the support point, and thus a performance of work is associated with the elevation of the figure axis. For friction can only dissipate and never provide work. In reality, of course, the energy required for the elevation of the center of gravity is drawn from the kinetic energy of the top, upon which the friction also feeds. The shortening of the impulse vector, which has as a consequence the diminishment of the *vis viva*, is always a necessary corollary to the uprighting of the impulse vector and the figure axis when the center of gravity lies above the support point.

The final result of sliding friction is thus the *upright motion of the top.* The impulse that remains available for this motion is given by the initial vertical component n of the impulse, through which the uniform rotational velocity of the upright motion is also predetermined. After the impulse axis, the rotation axis and the figure axis once become coincident in the vertical position, sliding friction does not occur: according to our present assumptions, the top can continue to rotate in this position without retardation for all time.

The latter result obviously contradicts the common experience that every motion will in the end be definitively annihilated by the influence of friction. In fact, this result is only a consequence of the arbitrary distinction between sliding and boring friction, and the omission of the latter. Thus we must now supply an approximate judgment of *the effect of boring friction.*

According to the provisional assumptions above, boring friction provides a moment that opposes the vertical component of the rotation vector. In the case of Fig. 77a (rotation in the clockwise sense about the figure axis), the rotation vector is directed upward, and therefore the turning-moment M_2 of the boring friction would be represented by an arrow that runs downward from O. This turning-moment, just like the turning-moment of the sliding friction, is composed at each instant with the instantaneous impulse OJ. It follows, as one sees, that the impulse vector *is deflected from the vertical*, in that its endpoint will be displaced somewhat from J to J_2.

The same also holds, however, for the case of Fig. 77b, where the arrow of the turning-moment M_2 would point upward and the impulse is elevated by the composition with M_2. In both cases, the change of the impulse due to the boring friction is associated with a *shortening of the impulse.*

Boring friction therefore acts, in one respect, in the opposite sense from that of sliding friction: *it strives to remove the impulse vector from the vertical.* In another respect, it acts in the same sense as the sliding friction: *it always weakens the impulse.* Since we saw that the effect of boring friction is small compared to the effect of sliding friction as long as the rotation axis is perceptibly different from the vertical, the latter will always tip the balance, and the uprighting of the figure axis will follow in spite of boring friction. At most, the time duration of the uprighting can be somewhat increased by the influence of boring friction. On the other hand, it is to be noted that the uprighting will occur faster as the impulse vector becomes shorter, and that boring friction will thus accelerate the uprighting indirectly, in that it reduces the length of the impulse vector. This indirect influence of the boring friction will compensate, in part, its direct effect of deflecting the impulse vector from the vertical.

If, however, the upright position is approximately attained, then boring friction itself is the main effect, since sliding friction then becomes

very small. The now vertical impulse will always be further weakened by boring friction, without at first changing the essential character of the upright motion. The rotational velocity of the upright motion, which can persist unchanged in the presence of sliding friction, will be decreased more and more by boring friction. If the center of gravity lies above the support point, the impulse must finally be reduced to the magnitude at which the upright motion becomes unstable. The smallest disturbance now generates a perceptible oscillation of the figure axis. This oscillation increases in amplitude for further decrease of the impulse, until the top falls, and, after a few apparently irregular last efforts, comes definitively to rest.

§4. Quantitative considerations on the influence of sliding friction on the inclination of the figure axis. Graphical integration of the corresponding differential equation.

The most appropriate basis for a thorough treatment of our friction problem is provided by the Lagrange equations in terms of the Euler angles φ, ψ, ϑ. In addition to the velocity coordinates φ', ψ', ϑ', we will make use of the impulse coordinates $[\Phi] = N$, $[\Psi] = n$, $[\Theta]$. In the transference of the symbol N, which previously signified a constant of integration, it is to be noted that this impulse component is now no longer constant, but rather is continually changed by sliding friction; with consideration of boring friction, the impulse component n would also become variable. If we disregard boring friction (Omission II), the forces that act on the top consist of gravity and sliding friction. Gravity causes a moment only about the line of nodes, and sliding friction, on the basis of our Omission III (see equations (10) of the previous section), produces a moment only about the figure axis. The coordinates of our external forces with respect to the three Euler angles are given in the following table.

	φ	ψ	ϑ
Gravity	0	0	$P\sin\vartheta$
Sliding friction	$\mp\varrho\mu Mg\sin\vartheta$	0	0

Here we have already made use of Omission I, in that the reaction has been identified with its static component Mg.

The fundamental equations from which we will begin have already been developed in an entirely analogous form on page 154 and pages 220ff.; they are

a) the expression

$$T = \frac{A}{2}\,(\sin^2\vartheta\,\psi'^2 + \vartheta'^2) + \frac{C}{2}\,(\varphi' + \cos\vartheta\,\psi')^2 \tag{1}$$

for the *vis viva* of the symmetric top;

b) the relations

$$\begin{cases} [\Phi] = N = \dfrac{\partial T}{\partial\varphi'} = C(\varphi' + \cos\vartheta\,\psi'), \\[2ex] [\Psi] = n = \dfrac{\partial T}{\partial\psi'} = A\sin^2\vartheta\,\psi' + C\cos\vartheta\,(\varphi' + \cos\vartheta\,\psi'), \\[2ex] [\Theta] \qquad = \dfrac{\partial T}{\partial\vartheta'} = A\vartheta' \end{cases} \tag{2}$$

between the impulse and the velocity coordinates;

c) the solutions

$$\psi' = \frac{n - \cos\vartheta\,N}{A\sin^2\vartheta}, \qquad \varphi' = \frac{N - \cos\vartheta\,n}{A\sin^2\vartheta} + \left(\frac{1}{C} - \frac{1}{A}\right)N \tag{3}$$

of the first two of the preceding equations for the velocity coordinates ψ' and φ';

d) the partial differential quotient

$$\frac{\partial T}{\partial\vartheta} = (A\cos\vartheta\,\psi' - N)\sin\vartheta\,\psi' = -\frac{(N - \cos\vartheta\,n)(n - \cos\vartheta\,N)}{A\sin^3\vartheta} \tag{4}$$

of the *vis viva* with respect to the coordinate ϑ;

e) the law of the impulse changes, or the Lagrange equations in the strict sense

$$\frac{dn}{dt} = 0, \tag{5}$$

$$\frac{dN}{dt} = \mp\varrho\mu M g\sin\vartheta, \tag{6}$$

$$A\vartheta'' + \frac{(N - \cos\vartheta\,n)(n - \cos\vartheta\,N)}{A\sin^3\vartheta} = P\sin\vartheta. \tag{7}$$

For the frictionless top, we used the theorem of the *vis viva* instead of equation (7), which was advisable since the execution of an integration was thus completed. In the present case this advantage is lost, since the frictional resistance is not a conservative force; thus equation (7) becomes, because of its simpler manner of construction, more convenient than the theorem of the *vis viva*.

We wish to review the meaning of the last three equations of the set.

Equation (5) states *that the vertical component of the impulse is not influenced by sliding friction*, as we have already recognized in the

previous section. Thus the quantity n can now, as previously, be regarded as a constant of integration that is determined by the initial state. This result follows, moreover, solely from our Omission II of the boring friction, and is independent of the introduction or nonintroduction of Omissions I and III.

We conclude from equation (6) *that the absolute value of the eigen-impulse N always changes in the same way, namely in a decreasing sense.* Because of the meaning of the double sign (cf. page 552), a negative or positive value for dN is calculated from equation (6) according to whether φ', or equivalently N, is positive or negative. We can thus use the magnitude of N as a kind of time measure, since we can refer the course of the motion to the decreasing value of $|N|$ just as well as to the increasing value of t. In other words, *we can introduce the magnitude of N as the independent variable instead of the time t.* If the changing position of the top, and in particular the angle ϑ, is known as a function of N, then the temporal course of the motion may be determined afterward, in that one calculates, according to (6),

$$t = \mp\frac{1}{Mg\mu\varrho}\int\frac{dN}{\sin\vartheta}. \tag{8}$$

Three variables are first apparent in equation (7); namely, t, N, and ϑ. Instead of ϑ, we introduce, as previously, the auxiliary quantity

$$u = \cos\vartheta. \tag{9}$$

We eliminate the variable t by means of equation (6), and subsequently use, according to the preceding remark, N as the independent variable. To this end, it is necessary only to replace the differential quotient of ϑ with respect to time by a differential quotient with respect to N. We have

$$\begin{cases} \dfrac{d\vartheta}{dt} = \dfrac{d\vartheta}{dN}\cdot\dfrac{dN}{dt} = \mp\varrho\mu Mg\sin\vartheta\dfrac{d\vartheta}{dN} = \pm\varrho\mu Mg\dfrac{du}{dN}, \\[2ex] \dfrac{d^2\vartheta}{dt^2} = \mp\varrho\mu Mg\dfrac{d^2u}{dN^2}\cdot\dfrac{dN}{dt} = -(\varrho\mu Mg)^2\dfrac{d^2u}{dN^2}\sin\vartheta. \end{cases} \tag{10}$$

Equation (7) may thus be written, with consideration of (9) and (10), in the noteworthy simple form

$$(\varrho\mu Mg)^2\frac{d^2u}{dN^2} = \frac{(N-un)(n-uN)}{A^2(1-u^2)^2} - \frac{P}{A}. \tag{11}$$

The problem is thus reduced to a single ordinary differential equation of the second order between u and N.

We do not intend to integrate this equation in closed form or by any

kind of series expansion. We will rather seek, without any formal integration, to discover the essential features of the course of the integral curves through an appropriate discussion of the differential equation.

Since the form of the integral curves depends essentially on their curvature, and since this curvature depends on the second differential quotient, we are led to study the right-hand side of (11) in more detail. And, in particular, we will first determine where the right-hand side changes in sign. For this purpose, we consider the equation

$$(n - uN)(N - un) - AP(1 - u^2)^2 = 0. \tag{12}$$

It is convenient here to divide by the square of the impulse constant n and to introduce the abbreviations

$$v = \frac{N}{n}, \quad \pm m^2 = \frac{AP}{n^2}. \tag{13}$$

The quantity v is then, just like the inclination cosine u, a pure number. The same holds, according to page 293, for the quantity $\pm m^2$. The positive or negative sign is chosen according to whether P is positive or negative, and thus whether the center of gravity lies above or below the support point. Our equation (12) is thus transformed into an equation between the three abstract numerical quantities u, v, and m^2; namely,

$$(1 - uv)(v - u) = \pm m^2(1 - u^2)^2. \tag{14}$$

We interpret u as the ordinate and v as the abscissa in a u,v-plane; we designate the curve of the fourth order in this plane represented by equation (14) as a *guideline*, since it will serve, to a certain extent, as a guide for the integral curves to be constructed below. The integral curves of (11) must, as we will show, oscillate around our guideline in its immediate vicinity.

The form of the guideline is represented in Fig. 78. It is depicted by the solid line for the case $P > 0$, where the positive sign in (14) obtains, and by the dotted line in the case $P < 0$, where m^2 is supplied with the negative sign. As equation (14) shows, the latter results from the former if one replaces u,v by $-u,-v$, or if one rotates the first line about the origin through the angle 180°. Thus it suffices to consider only the case $P > 0$, and to consider only the upper sign in equation (14).

As a justification of our figure, it is noted that if one constructs the equilateral hyperbola $1 = uv$ and the line $v = u$, then the u,v-plane is divided into six regions; in three of these regions the left-hand side of (14) is positive, and in the remaining three it is negative. Our curve can

run only in the former regions, which are made recognizable in the figure by hatching, since otherwise equation (14) would not be fulfilled.

Furthermore, it is essential for the form of our guideline that we may assume m^2 to be a *small number.* For we consider only motions in

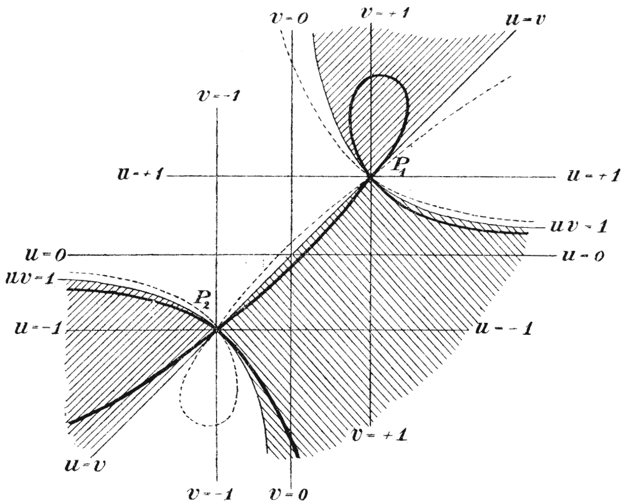

Fig. 78.

which the top is initially given a rapid rotation or a strong impulse. By a strong impulse we understand, according to page 293, an impulse for which N^2 is considerably larger than the equidimensional quantity AP, and therefore for which AP/N^2 is a small proper fraction $\left(\text{for example, } < \dfrac{1}{100}\right)$. Since the vertical component n of the impulse is of the same order of magnitude as the initial value of the eigenimpulse, $AP/n^2 = m^2$ will also be a small proper fraction. In the figure we have chosen $m^2 = 1/9$, since for smaller m^2 our drawing would be unclear, while we will stipulate for the purpose of later calculations that $m^2 < \dfrac{1}{100}$.

One is then easily convinced, according to the usual method of power series expansion, that the points P_1 $(u = v = 1)$ and P_2 $(u = v = -1)$ are double points of our curve, and that the two tangents to the curve at these points enclose an angle α with respect to the positive or negative axis of the abscissa, which is calculated from the equation

$$\operatorname{tg}\alpha = \sqrt{\frac{1}{1-4m^2}} \quad (\text{Point } P_1)$$

or

$$\operatorname{tg}\alpha = \sqrt{\frac{1}{1+4m^2}} \quad \text{(Point } P_2\text{)}.$$

At point P_1, the named angle is therefore slightly greater than 45°, and at P_2 slightly less than 45°.

It further follows easily from equation (14) that our guideline possesses one and only one tangent parallel to the v-axis, with the tangent point

$$u = \frac{1}{4m^2}, \quad v = \frac{1}{2}\left(4m^2 + \frac{1}{4m^2}\right).$$

Thus it is to be concluded that the two upward-running branches of the curve through P_1 join in a loop. The two branches running to the left through P_2, in contrast, cannot come together, since the upper of them approaches the abscissa axis asymptotically. The same holds for the branch through P_1 running downward and to the right.

Through this and similar considerations, the form of our guideline is established with sufficient certainty in the case $P > 0$. Its form in the case $P < 0$ is then derived by the previously mentioned rotation.

What use does the knowledge of the guideline now afford us for the integration of equation (11)? We first write this equation so that only dimensionless quantities appear. To this end, we divide it by n^2/A^2 and replace

$$\frac{d^2u}{dN^2} \text{ by } \frac{1}{n^2}\frac{d^2u}{dv^2}.$$

If we introduce the usual abbreviation $P = \pm MgE$ and the further abbreviation

$$\lambda = \frac{\varrho}{E},$$

then the coefficient of $\dfrac{d^2u}{dv^2}$ becomes

$$\frac{A^2}{n^4}(Mg\mu\varrho)^2 = \left(\frac{AP}{n^2}\mu\frac{\varrho}{E}\right)^2 = (m^2\mu\lambda)^2,$$

and our equation (11) goes over into

$$(15) \qquad (m^2\mu\lambda)^2\frac{d^2u}{dv^2} = \frac{(1-uv)(v-u)}{(1-u^2)^2} - m^2 \;\cdots\; (P>0)$$

or

$$(16) \qquad (m^2\mu\lambda)^2\frac{d^2u}{dv^2} = \frac{(1-uv)(v-u)}{(1-u^2)^2} + m^2 \;\cdots\; (P<0).$$

Now the right-hand side of this equation vanishes only at the points of the corresponding guideline, and thus a change in the sense of the curvature of our integral curve occurs only if it crosses the guideline.

Because of the meaning of $u = \cos\vartheta$, we need only consider the strip of the u,v-plane that is contained between the lines $u = \pm 1$; this strip is divided by the guideline into four regions. The signs of d^2u/dv^2 in each region are registered in Figs. 79 and 80; one establishes them most simply if one begins from the point $u = v = 0$, at which the right-hand side of (15) is equal to $-m^2$, and that of (16) is equal to $+m^2$. Thus the relevant sign for each point in our strip is determined. *In the regions designated with the + sign, the desired integral curve is concave as viewed from the direction of the positive ordinate axis, and in the regions designated with the − sign it is convex; it possesses a point of inflection each time it crosses the guideline.*

To be able to actually construct the integral curve, we must first give specific initial conditions. We designate the initial inclination cosine of the figure axis with respect to the vertical by u_0, and stipulate, for example, that the impulse vector initially falls exactly onto the direction of the figure axis, so that the top receives no initial lateral impulse. Then the initial value N_0 of the eigenimpulse gives, at the same time, the total length of the impulse vector, and the vertical component of the impulse is $n = N_0 u_0$. Our integral curve thus begins at a point P_0 whose coordinates u_0, v_0 satisfy the equation $1 = u_0 v_0$, and therefore lies on the equilateral hyperbola that is drawn in dashed lines in Figs. 79 and 80. The initial tangent to the integral curve is also now determined; if, namely, the impulse vector has the direction of the figure axis, then the instantaneous rotation axis also falls onto the figure axis. The figure axis is therefore instantaneously at rest in space, and so

$$\frac{du}{dt} = 0, \quad \text{and thus} \quad \frac{du}{dN} = 0 \quad \text{and} \quad \frac{du}{dv} = 0.$$

Our integral curve therefore touches the point P_0 with a horizontal tangent.

The further course of the integral curve satisfies the general remark *that the projection of the integral curve onto the axis of the abscissa must cover this entire axis simply.* For, as established above, the absolute value of N always decreases with increasing t, and likewise the (necessarily positive) value of $v = \dfrac{N}{n}$. Now since only one value of u can correspond to each value of t, each value of v can also correspond to only one value of u.

We now consider, for example, Fig. 79 ($P > 0$). The initial point P_0 lies in a region of negative curvature (a region where $d^2u/dv^2 < 0$);

the integral curve is therefore convex as seen from above. Since it has a horizontal tangent at P_0, it must bend downward and soon intersect the guideline. It then goes over with a point of inflection into a region with positive curvature, and therefore becomes concave upward. The

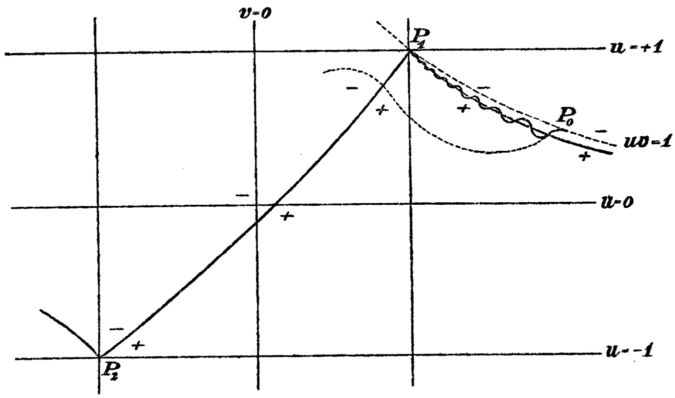

Fig. 79.

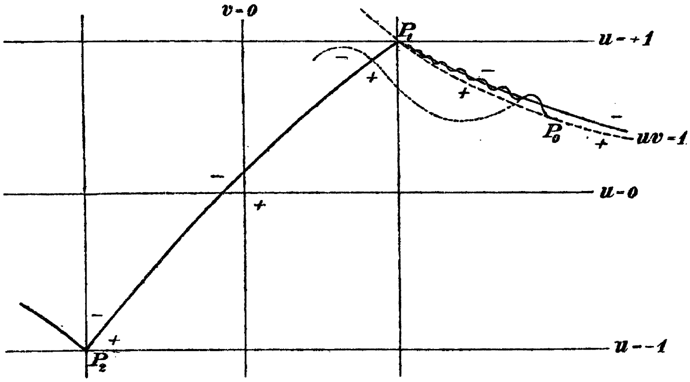

Fig. 80.

two possibilities that are now offered are indicated in Fig. 79. The integral curve must intersect the guideline a second time; this intersection point can lie between P_0 and P_1 or beyond P_1. The wavy solid form of the integral curve drawn in the figure corresponds to the first possibility, and the dotted form corresponds to the second. *We will show that only the first possibility corresponds to reality.*

For a proof, we must consider the magnitude of the curvature in addition to its sign. The former is given in rectangular coordinates u,v by the well-known expression

$$\text{a)}\qquad \frac{\dfrac{d^2u}{dv^2}}{\left(1+\left(\frac{du}{dv}\right)^2\right)^{3/2}}.$$

We will substitute for the curvature the approximate expression

$$\text{b)}\qquad \frac{d^2u}{dv^2},$$

whose instantaneous magnitude can be taken directly from the differential equation (19). This value is indeed somewhat too large, and coincides sufficiently well with the exact value of the curvature only if the inclination of the tangent to the curve with respect to the axis of the abscissa is small. We cannot claim with certainty that this occurs in our case; it is only clear from a remark on the previous page that the inclination of the tangent to the curve never becomes infinitely large, for then the projection of the integral curve onto the axis of the abscissa would not cover this axis in a single-valued manner.

On the guideline itself, as we know, the integral curve has curvature zero. If we replace the number m^2 in equation (15) by a slightly smaller or larger number, then there arise two neighboring curves of the guideline with essentially the same course, both of which pass through the point P_1 and asymptotically approach the positive axis of the abscissa. We can, for example, replace the small number m^2 by zero in one case and by $2m^2$ in the second case. The first of our neighboring curves then coincides with the hyperbola $uv = 1$ in the region of interest, and the second has the equation

$$(1-uv)(v-u) = 2m^2(1-u^2)^2.$$

At the points of the first or the second neighboring curve, the integral curve has, according to equation (15), the approximate curvature

$$\frac{d^2u}{dv^2} = \frac{-m^2}{(m^2\mu\lambda)^2} = -\frac{1}{m^2\mu^2\lambda^2}$$

or

$$\frac{d^2u}{dv^2} = \frac{2m^2-m^2}{(m^2\mu\lambda)^2} = +\frac{1}{m^2\mu^2\lambda^2}.$$

The value of m^2 should be, for example, $\frac{1}{100}$. The friction coefficient μ is a proper fraction; as its order of magnitude one can assume, say, $\frac{1}{3}$, so that μ^2 becomes about $\frac{1}{10}$. The ratio $\lambda = \frac{\varrho}{E}$ is also a proper fraction,

since the center of gravity must have a perceptible distance from the fixed point if we are actually considering a “heavy top,” while the hemisphere that bounds the lower end of the figure axis certainly has a small radius. In order to make definite a specification, we wish to set, for example, λ^2 equal to $\frac{1}{1000}$ (that is, $E =$ ca. 32ϱ). Under this assumption, the approximate curvature of the integral curve on our two curves neighboring the guideline is equal to $\pm 10^6$, and the approximate radius of curvature is therefore equal to only one millionth of the unit length of our figure. The distance of our two neighboring curves from each other and from the guideline is also extremely small; it is, namely, of the order of magnitude of m^2 itself, and it diminishes, moreover, with increasing approach to the point P_1.

Thus the curvature of the integral curve, which has the value zero on the guideline itself, will become very large in the nearest neighborhood of the guideline. As soon as the integral curve is removed only perceptibly from the guideline, it must rapidly bend and approach the guideline once more. The integral curve is thus forced to oscillate about the guideline with extremely small amplitude and span width, similar to a mass particle that oscillates about an equilibrium position with a small amplitude and a small oscillation period, as if it were pushed back by a large force for a small removal from the equilibrium position.

Thus it is proven that the dotted course of the integral curve in Fig. 79 is impossible for small values of m^2 (that is, for a large initial impulse), and that the serpentine solid curve corresponds at least qualitatively to reality. The dotted path may perhaps become valid for a weak initial impulse, but we do not enter into this less important case. The corresponding deliberation and construction can be carried over almost word for word to the case $P < 0$; we can thus claim that the integral curve in Fig. 80 must also oscillate about the guideline and can never significantly turn away from it.[211]

Moreover, the manner of conclusion followed here, which we can designate as graphical integration, may be carried over immediately to the general case of the differential equation

$$\frac{d^2u}{dv^2} = f(u, v)$$

if the function $f(u, v)$ has a large gradient in the neighborhood of the “guideline” $f(u, v) = 0$ and the initial point of the integral curve is taken

not too far from the guideline. Here too the integral curve must progress by oscillating about the guideline.

Concerning the amplitude and span width of the oscillation, we have said until now only that they must be extremely small; we add that they *must always become smaller as the number* m^2 *is smaller, and therefore as the initial impulse of the motion is larger, and that they must decrease with increasing approach to the point* P_1.

To visualize this, we imagine constructing the level lines of the expression $f(u, v)$, which, in consequence of our differential equation, determine the approximate curvature of the integral curve, as was done for the special level line $f(u, v) = 0$ (the guideline) and $f(u, v) = \pm\dfrac{1}{(m\mu\lambda)^2}$ (the two neighboring curves named above). These level lines are always denser as m^2 is smaller, and are extremely compact in the vicinity of the point P_1, since they must all pass through this point. The density of the level lines directly provides, however, a measure of the curvature increase of the integral curve in the vicinity of the guideline, and of its tendency to come back toward the guideline. Still more intuitively, we could imagine the expression $f(u, v)$ modeled as a relief, in that we lay off the absolute value of $f(u, v)$ as the third coordinate perpendicular to the u,v-plane, so that the level lines become elevation lines of the relief. A channel is thus formed, whose bottom lies in the u,v-plane and coincides with the guideline, and whose banks on both sides rise ever more steeply as m is smaller and as we approach the point P_1. At this point the banks are exactly vertical. The rapidity with which the integral curve returns to the guideline after a lateral deflection increases with the steepness of the banks. The integral curve is thus analogous to the trajectory of a particle that runs along the (assumed frictionless) channel, rising somewhat onto the right and left banks due to a lateral impact. While the equal heights of the successive lateral oscillations are determined by the energy law, the amplitude of the lateral deviations in the horizontal direction will always be smaller as the steepness of the sides is greater; at the same time, the duration of the sequential oscillations, or, equivalently, the length of the span width measured on the bottom, decreases with increasing steepness of the sides; for the restoring force toward the channel—that is, the component of gravity along the bank—is proportional to the gradient of the bank.

For what concerns the extent of the sequential oscillations, the horizontal projection of the path of the mass particle will in fact take the form of our integral curve represented in Figs. 79 and 80.

The final results for the course of the motion of the top are now at hand. With increasing time, the eigenimpulse N decreases in magnitude. If the eigenimpulse initially falls in the direction of the figure axis, then $|N| > |n|$ initially, and, with increasing time, N approaches the value n; that is, v approaches the value 1. Our integral curve then shows that u simultaneously approaches the value 1, or ϑ approaches the value 0. *Thus the figure axis is gradually uprighted by the effect of sliding friction.*

The uprighting of the figure axis naturally occurs hand in hand with its *precession* about the vertical, whose instantaneous velocity, according to eqn. (3), is computed from the instantaneous values of ϑ and N, or of u and v. The uprighting of the figure axis will be interrupted and its precession will be accompanied by a small *nutation* of the figure axis, which is represented by the lateral oscillations of our integral curve. *This nutation expires, however, as the figure axis is uprighted, and is from the beginning, moreover, always smaller as the initial impulse is greater*, naturally assuming that this initial impulse has, exactly or approximately, the direction of the figure axis.

If the upright position is attained, then the present basis for the decrease of the impulse, the sliding friction, is eliminated. In fact, equation (6) implies that $dN/dt = 0$ when $u = 1$; thus $N = n$ or $v = 1$ forever more. *Our integral curve ends at the point P_1, and the top remains in upright motion.* The final destruction of the impulse of the motion is due not to sliding friction, but rather to boring friction, as was already discussed in the previous section.

§5. Approximate formulaic representation of the course of the motion.

Since we command the motion of the figure axis graphically on the basis of the preceding discussion, it will now be easy to give an approximate formulaic representation of the motion. We add this afterward, in part to be able to present some numerical calculations, and in part to fulfill the precept stated in the introduction (page 5), according to which “our knowledge of mechanics should not be based on formulas, but rather, on the contrary, the analytic formulation should appear

of itself as the final consequence of a fundamental understanding of the mechanical principles."

The idea for the following approximate calculations is that we replace, for what concerns the change of the angle ϑ, the oscillating integral curves of Figs. 79 and 80 by our guideline itself. We thus neglect the nutation of the figure axis, which influences the motion only in passing and to a small degree, but retain and simply express in our formula the uprighting of the figure axis, the diminishment of the impulse vector, and the mean value of the precession; that is, all the essential properties of the motion.

We thus regard equation (14) of the previous paragraph as an approximately valid relation between the inclination cosine $u = \cos\vartheta$ and the impulse quantity $v = \dfrac{N}{n}$ during the course of the motion. To solve this relation for v, we write it as

$$v^2 - \left(u + \frac{1}{u}\right)v = -1 \mp \frac{m^2}{u}(1-u^2)^2.$$

The two roots v_1, v_2 of this quadratic equation are

$$v_1 = \frac{1}{2}\left(u + \frac{1}{u}\right) - \frac{1}{2}\left(u - \frac{1}{u}\right)\sqrt{1 \mp 4um^2},$$
$$v_2 = \frac{1}{2}\left(u + \frac{1}{u}\right) + \frac{1}{2}\left(u - \frac{1}{u}\right)\sqrt{1 \mp 4um^2}.$$

Because of the assumed smallness of the number $\pm\, m^2 = \dfrac{AP}{n^2}$, we extract the square root approximately according to the binomial theorem. There follows

$$v_1 = \frac{1}{2}\left(u + \frac{1}{u}\right) - \frac{1}{2}\left(u - \frac{1}{u}\right)(1 \mp 2um^2) = \frac{1}{u} \mp m^2(1-u^2),$$
$$v_2 = \frac{1}{2}\left(u + \frac{1}{u}\right) + \frac{1}{2}\left(u - \frac{1}{u}\right)(1 \mp 2um^2) = u \pm m^2(1-u^2).$$

Since $u < 1$, $v_1 > 1$ and $v_2 < 1$. The meaning of the two roots follows from Fig. 78. If we cut the solid or the dotted guideline of this figure with a line $u =$ const. parallel to the axis of the abscissa, where $0 < u < 1$, then we obtain two intersection points, of which one lies to the right of P_1 and the other to the left, and thus between P_1 and P_2. The former corresponds to an abscissa value $v_1 > 1$, and the latter to a value $v_2 < 1$. We are interested only in the part of the guideline about which our integral curve oscillates, and therefore have only the root v_1 to consider.

If we return to the original meanings of the symbols v, u, and m^2, then we can write the formula for v_1 as

$$N = \frac{n}{\cos\vartheta} - \frac{AP}{n}\sin^2\vartheta. \tag{1}$$

We thus recognize *the mutual dependence in which ϑ converges to* 0 *and N converges to n.*

We calculate secondly the precessional velocity ψ' that corresponds to the changing inclination of the figure axis. From (1) there follows

$$\frac{n - N\cos\vartheta}{A\sin^2\vartheta} = \frac{P}{n}\cos\vartheta.$$

This is also, according to equation (3) of the preceding section, the desired precessional velocity. One thus has

$$\psi' = \frac{P}{n}\cos\vartheta, \tag{2}$$

and concludes *that the absolute value of the precessional velocity accelerates somewhat in the uprighting of the figure axis.*

We next ask for the temporal course of the motion, which indeed was eliminated from our qualitative representation. We must therefore return to equation (8) of the preceding section,

$$t = \mp\frac{1}{Mg\mu\varrho}\int\frac{dN}{\sin\vartheta}\,.$$

(The upper sign obtains for positive initial values of N, and thus for positive n, and the lower sign for negative values.) We calculate dN in terms of ϑ and $d\vartheta$ from equation (1) as

$$dN = \left(\frac{n}{\cos^2\vartheta} - \frac{2AP}{n}\cos\vartheta\right)\sin\vartheta\,d\vartheta,$$

and thus obtain

$$t = \mp\frac{n}{Mg\mu\varrho}\left\{\int\frac{d\vartheta}{\cos^2\vartheta} - \frac{2AP}{n^2}\int\cos\vartheta\,d\vartheta\right\}.$$

We may replace the double sign by a simple negative if we provide n with its absolute value. If we carry out the integration and determine the constant of integration so that $\vartheta = \vartheta_0$ at $t = 0$, there follows the *law for the temporal course of the motion*:

$$t = \frac{|n|}{Mg\mu\varrho}\left\{(\operatorname{tg}\vartheta_0 - \operatorname{tg}\vartheta) - \frac{2AP}{n^2}(\sin\vartheta_0 - \sin\vartheta)\right\}. \tag{3}$$

The second term in the braces { } is evidently small, due to the factor AP/n^2, compared to the first term. This first term shows us that the uprighting of the figure axis takes place rather slowly; for in the numerator

stands the magnitude of the impulse component n, and in the denominator the small friction coefficient μ and the small radius ϱ of the support surface. *The time duration of the uprighting is always greater as the initial impulse is greater and as the friction coefficient μ and the radius of curvature ϱ of the bearing surface are smaller.*

The numerical value of the time required for the uprighting follows from (3), if we set $\vartheta = 0$, as

$$T = \frac{|n|}{Mg\mu\varrho}\Big\{\operatorname{tg}\vartheta_0 - \frac{2AP}{n^2}\sin\vartheta_0\Big\}. \tag{4}$$

The upright position of the top will therefore be attained in a finite time; this time is essentially proportional, for otherwise equal circumstances, to the tangent of the initial inclination.

It now remains only to represent, analytically and graphically, the trajectory that is described by a point on the figure axis. We begin, on the one hand, from the equation

$$\frac{d\psi}{dt} = \frac{P}{n}\cos\vartheta$$

given by (2), and, on the other hand, the relation

$$\frac{d\vartheta}{dt} = \mp\frac{Mg\mu\varrho}{n}\frac{\cos^2\vartheta}{1 - \dfrac{2AP}{n^2}\cos^3\vartheta} \tag{5}$$

that follows from (3). By division there follows

$$\frac{d\psi}{d\vartheta} = \pm\frac{P}{Mg\mu\varrho}\Big(\frac{1}{\cos\vartheta} - \frac{2AP}{n^2}\cos^2\vartheta\Big),$$

and by integration

$$\psi = \mp\frac{P}{Mg\mu\varrho}\Big\{\log\operatorname{ctg}\Big(\frac{\pi}{4} - \frac{\vartheta}{2}\Big) - \frac{AP}{n^2}\Big(\vartheta + \frac{\sin 2\vartheta}{2}\Big)\Big\}. \tag{6}$$

Since the form of the trajectory depends in no way on the initial value ψ_0 prescribed for the angle ψ, we have disregarded the addition of a constant of integration.

Here we wish to allow an inessential omission through which the following is simplified. We wish, namely, to strike the second term in the braces $\{\quad\}$ in (6) in comparison with the first, because of the factor AP/n^2. We further wish, in order to have a determined sign, to temporarily assume that the center of gravity lies *above* the support point, and that the initial impulse vector has the approximate direction of the *positive* figure axis. Then $P = +MgE$ is to be set, and the upper sign in (6) is to be chosen. If we introduce the already used ratio $\lambda = \varrho/E$, then (6) can be written as

$$\lambda\mu\psi = \log \operatorname{tg}\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right),$$

or also

$$e^{\lambda\mu\psi} = \operatorname{tg}\left(\frac{\pi}{4} - \frac{\vartheta}{2}\right) = \frac{1 - \operatorname{tg}\vartheta/2}{1 + \operatorname{tg}\vartheta/2},$$

or finally

$$\operatorname{tg}\vartheta/2 = \frac{1 - e^{\lambda\mu\psi}}{1 + e^{\lambda\mu\psi}}. \tag{7}$$

This is the desired equation of the trajectory for positive P and positive initial impulse. It obviously holds just as well for other choices of the signs of P and n, if only one reverses, if necessary, the sense in which ψ is calculated.

In order to draw the trajectory, we must somehow project it onto a drawing plane. It is recommended here, as previously, to adopt the *stereographic projection.* We thus place a unit sphere about the fixed point O, on which our trajectory would run if the point that generated it had distance 1 from O, and project from the south pole of the unit sphere to the equatorial plane. The north pole thus goes over into the point O, while the image of any other point of the unit sphere has distance $r = \operatorname{tg}\vartheta/2$ from O and azimuth ψ. The quantities r and ψ are therefore the usual polar coordinates of the stereographic image point, with point O as the origin. In these coordinates, the image of the trajectory is described, according to (7), by

$$r = \frac{1 - e^{\lambda\mu\psi}}{1 + e^{\lambda\mu\psi}}. \tag{8}$$

Its form is thus a *spiral*; it runs into the point O as an ordinary *spiral of Archimedes*, while, in the other direction, it approaches *the unit circle asymptotically.*

To see this, one notices that, due to the value of the constant of integration in equation (6), the upright final position ($\vartheta = 0$ or $r = 0$) corresponds to the azimuth $\psi = 0$, and that all previous positions of the figure axis correspond to negative values of ψ. To investigate the behavior of the trajectory in the vicinity of the point O, we must therefore assume ψ to be small, and expand the exponential function in powers of $\lambda\mu\psi$. There follows

$$r = -\lambda\mu\,\frac{\psi}{2}; \tag{8'}$$

that is, the equation of a spiral of Archimedes.

To determine the trajectory for times far in the past, on the other hand, we must give ψ a large negative value, and therefore regard $e^{\lambda\mu\psi}$

as small. The radius r then tends to the upper limit 1, and the trajectory thus tends asymptotically to the unit circle.

In the vicinity of O, the decrease of the radius r for a complete circuit about O is given by $\lambda\mu\pi$; this is small, since λ and μ are small numbers, but does not vanish in the approach to O. If, on the other hand, we follow the curve in reverse to the vicinity of the unit circle, then the increase of the radius for a single circuit about O obviously becomes vanishingly small with increasing approach to the unit circle.

One can establish the form of the trajectory very beautifully by experiment if one allows the apex of the top to trace its path on a surface that remains at rest, or, what is still more to be recommended, if one fixes a small mirror perpendicular to the axis of the top, illuminates the mirror with a projection apparatus, and observes the reflected spot of light on a screen. The curve obtained in this manner has entirely the character of the spiral depicted here, except that the uniform course of the spiral is broken by a superposed oscillation (nutation) that we have neglected in our representation.

Figure 81 is drawn under the assumption $\lambda\mu = 1/10$. It corresponds poorly to reality, since the actual value $\lambda\mu$ is generally much smaller. In a previous place (page 565), we estimated $\mu^2 = 10^{-1}$, $\lambda^2 = 10^{-3}$; therefore $\lambda\mu = 1/100$ would be closer to reality. On the basis of this value, however, the drawing would become unclear.

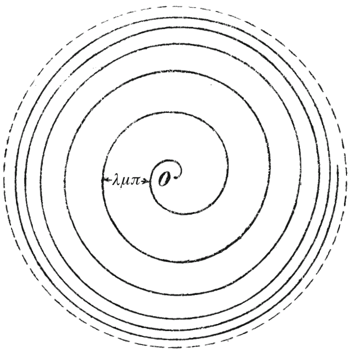

Fig. 81.

For the following numerical calculations, on the the other hand, we wish to use the latterly given value. We first ask how many windings the trajectory executes, for a given initial position, before it ends at the origin. Let the initial position be, for example, $\vartheta_0 = 60°$. We take the corresponding value of the azimuth ψ_0, which must be negative, from equation (8), or, somewhat more precisely, from equation (6), in that we calculate with the previously neglected term of this equation; ψ_0 becomes

$$\psi_0 = -\frac{1}{\lambda\mu}\left\{\log\operatorname{ctg}\left(\frac{\pi}{4} - \frac{\vartheta_0}{2}\right) - \frac{AP}{n^2}\left(\vartheta_0 + \frac{\sin 2\vartheta_0}{2}\right)\right\}.$$

With $\lambda\mu = 10^{-2}$, $\vartheta_0 = 60° = 1{,}05$, and the previously assumed value $\dfrac{AP}{n^2} = 1/100$, there follows

$$\psi_0 = -130{,}2.$$

This is the *initial value* of the angle ψ. Since the *final value*, for the upright position of the figure axis, is $\psi = 0$, ψ_0 gives us, at the same time, the total angle through which the radius from O has turned in the motion. The number of windings of the spiral is therefore

$$\frac{|\psi_0|}{2\pi} = 21.$$

The figure axis therefore encircles the vertical a considerable number of times before it coincides with it. The pitch of the trajectory will thus be quite small in the stereographic projection, and considerably smaller than in the case of Fig. 81, where the corresponding number of circuits is only 2,1.

We can determine the approximate frequency of the nutation retroactively, although it has been removed from our consideration. It may be shown that the period τ of the nutation has approximately the same value as for the assumed frictionless regular precession; namely (see equation (15) of page 305), the value

$$\tau = \frac{2\pi A}{N}. \tag{9}$$

For a proof, we return to the differential equation (7) of page 558, set there $\vartheta = \vartheta_1 + \vartheta_2$, and understand by ϑ_1 the previously studied precession-like component of the motion and by ϑ_2 the additional nutation. The quantity ϑ_1 is thus a *slowly varying quantity*, and ϑ_2 is a *rapidly varying but small quantity.* We will correspondingly neglect ϑ_1'' with respect to ϑ_2'', and retain only the first power of ϑ_2 in the expansion of equation (7) in terms of ϑ_2. There follows, with consideration of the definition of ϑ_1 from the equation of the guideline,

$$A\vartheta_2'' + \vartheta_2 \cdot \frac{\partial}{\partial \vartheta_1} \frac{(N - \cos\vartheta_1 n)(n - \cos\vartheta_1 N)}{A \sin^3 \vartheta_1} = 0.$$

The differentiation indicated here gives simply N^2/A (cf. §9, equation (13), where an analogous calculation is carried out). The determining equation for ϑ_2 is thus $\vartheta_2'' + \vartheta_2 \dfrac{N^2}{A^2} = 0$, which yields, after integration, the period given above.

If T, on the other hand, denotes the time duration of an individual precessional circuit, and one disregards the small acceleration of the precessional velocity caused by the uprighting of the figure axis, then one can set

$$\frac{2\pi}{\mathsf{T}} = \psi',$$

and thus, according to equation (2),

$$\frac{2\pi}{\mathsf{T}} = \frac{P}{n}\cos\vartheta. \tag{10}$$

From (9) and (10) there follows, with consideration of (1),

$$\frac{\mathsf{T}}{\tau} = \frac{nN}{AP}\frac{1}{\cos\vartheta} = \frac{n^2}{AP}\frac{1}{\cos^2\vartheta}\left(1 - \frac{AP}{n^2}\cos\vartheta\sin\vartheta\right).$$

Here ϑ is understood as a mean value of the inclination angle ϑ during the relevant precessional circuit. The ratio T/τ signifies the *number of nutations that occur during one precession.* This number is, as we see, of the same order of magnitude as n^2/AP, and therefore, under the above numerical assumption, equal to 100. *The oscillations superposed on our spiral are therefore very frequent and dense.*

Finally, we ask for the numerical value of the time duration T during which the figure axis is uprighted. We express this, for example, in units of the rotation time τ_0 of the top after the completed uprighting. The total impulse is then exactly n, and the length of the rotation vector is $|n|/C$. The time duration τ_0 thus follows as

$$\frac{2\pi}{\tau_0} = \frac{|n|}{C}.$$

If we multiply this equation by equation (4), we obtain

$$\frac{T}{\tau_0} = \frac{1}{2\pi}\frac{A}{C}\frac{n^2}{AP}\frac{1}{\lambda\mu}\left\{\operatorname{tg}\vartheta_0 - \frac{2AP}{n^2}\sin\vartheta_0\right\}.$$

With $n^2/AP = 100$, $\lambda\mu = 1/100$, and $\vartheta_0 = 60°$, there follows

$$\frac{T}{\tau_0} = \frac{A}{C}2730.$$

If the top makes five rotations per second after the upright position is attained, then $\tau_0 = 1/5$ sec., and if one assumes as a special case that $A = C$, then $T = 546$ sec. = ca. 10 minutes.

It is to be kept in mind in our last calculation and in the description of the trajectory, however, that our considerations apply only up to a neighborhood of the upright position, and not to this position itself. For we have (Omission II) neglected boring friction with respect to sliding friction, which is permissible only for a not too small angle ϑ (cf. page 551). We must therefore regard the last part of our trajectory near the point O as unauthenticated.

We wish, finally, to classify the motion considered here in the system of the frictionless motions, in that we place $\mu = 0$ in our formula. We thus begin from equation (5), which with $\mu = 0$ yields

$$\vartheta' = 0, \quad \text{or} \quad \vartheta = \text{const.}$$

This is, at the same time, the equation of the trajectory for vanishing friction. From equations (1) and (2) it now follows that N and ψ' also become constant. *For vanishing friction, the motion considered here thus goes over into regular precession.* We can therefore denote it as "precession-like," or as a "precession damped by friction." If we include the nutation in the calculation, which is not expressed in our formulas, but which, as we saw in the preceding paragraphs, is superposed on our motion, then our current consideration is directly related to the previous investigation of *pseudoregular precession*, and shows us how this most important frictionless motion is modified by friction.

§6. On an obvious error in the assumptions of the friction problem. Subsequent rectification of the previous treatment and suggestions for experiments.

The primary purpose of the present section is to correct or restrict our earlier results on the determination of the frictional work and the frictional moments (cf. §3, page 550). We thus discuss a certain characteristic difference between friction forces, or, more generally, forces that depend in magnitude and direction on the velocity of the system, and forces that are determined in magnitude and direction by the instantaneous position alone, which have been most commonly considered in the development of theoretical mechanics.

On page 550, we calculated separately the work of an infinitesimal rotation about a horizontal axis and the work of an infinitesimal rotation about a vertical axis, and spoke of the former as the work of sliding friction and the latter as the work of boring friction. In formulas, we wrote

$$(1)\qquad \begin{cases} d\mathfrak{A}_1 = -\mu R\,\varrho\,\Omega\sin\alpha\,dt \\ d\mathfrak{A}_2 = -\mu' R\,\Omega\cos\alpha\,dt = -\mu R a\,\Omega\cos\alpha\,dt, \end{cases}$$

where a signified the order of magnitude of the radius of the contact circle. The total frictional work performed in an infinitesimal rotation $\Omega\,dt$ about an axis inclined to the vertical by the angle α was thus

$$(2)\qquad d\mathfrak{A} = -\mu R\,\Omega\,(\varrho\sin\alpha + a\cos\alpha)\,dt.$$

Is this procedure permissible without further consideration?

We first wish to consider the simple case of a single mass particle that moves in a plane. In one case the particle moves under the influence of

a force P that is determined only by the position of the particle, and in a second case it moves under the influence of a frictional force. If we adopt the Coulomb law of friction, then the magnitude of the friction force W is indeed independent of the velocity; it is equal, namely, to μR, where R signifies the reaction of our plane on the point. The direction of the friction force, however, depends on the velocity; the force is opposed, namely, to the instantaneous direction of the velocity. The work on the path increment ds in the first case is

$$d\mathfrak{A} = P\cos(P, ds)\, ds, \tag{3}$$

and in the second case is

$$d\mathfrak{A} = -W ds = -\mu R\, ds. \tag{4}$$

Alternatively, we calculate these two quantities of work by decomposing the path ds into two rectangular components dx and dy. On the path dx, P performs the work $P_x\, dx$, where P_x denotes the component of P with respect to the x-axis. The work on the path dy is calculated correspondingly; the total work is thus

$$d\mathfrak{A} = P_x\, dx + P_y\, dy, \tag{3'}$$

which, as is well known, coincides with (3).

If we would proceed in the same manner for the second case, we would say that if we first executed the motion dx, then the work of W on this path would equal $-W\, dx = -\mu R\, dx$. For in the motion dx, the friction force is opposite to the sense of the motion, and is therefore in the direction of the negative x-axis; it is given in magnitude by the coefficient of friction and the reaction force R. Similarly, the work on the path dy would equal $-W\, dy$. In total, one would thus obtain

$$d\mathfrak{A} = -W(dx + dy) = -\mu R(dx + dy), \tag{4'}$$

which evidently does not coincide with (4).

The calculation of the frictional work from the work of the component motions is thus, executed in this way, inadmissible. It is easy to see, however, how one must correct this calculation if the motion is decomposed into the components dx and dy: one must decompose the friction force W for the actual motion into the two components $W_x = W\dfrac{dx}{ds}$ and $W_y = W\dfrac{dy}{ds}$, and determine the work of these component forces in the component motions dx and dy. There then follows correctly, and in conformity with (4),

$$d\mathfrak{A} = -(W_x\, dx + W_y\, dy) = -\mu R\,\frac{dx^2 + dy^2}{ds}.$$

For friction forces, and, more generally, for forces that depend in any way on velocity, one must always similarly distinguish *between the work*

that would be done if the actual motion were separated into components and these components were considered as if they existed separately, and the work done in the imagined component motions by the forces that occur in the actual motion. *) One must adopt the second manner of calculating the work in the formation of the equations of motion; the first would be misleading.

In §3, however, this distinction was not made in the construction of the expressions that are repeated in (1). The work $d\mathfrak{A}_1$ done in the rotation $\Omega \sin\alpha\, dt$ about a horizontal axis and the work $d\mathfrak{A}_2$ done in the rotation $\Omega \cos\alpha\, dt$ about a vertical axis were calculated separately, as if one or the other of the two rotations were present alone; it was tacitly assumed that the work $d\mathfrak{A}$ performed in the rotation $\Omega\, dt$ about an arbitrarily inclined axis is composed additively from the work quantities $d\mathfrak{A}_1$ and $d\mathfrak{A}_2$. This is, according to the example above, incorrect; we must therefore confirm the calculation of the work $d\mathfrak{A}$ after the fact.

In order to be able to reduce boring friction to sliding friction, we do not contract into a point the contact circle between the spherical end of the figure axis and the seat that bears the top. There then arises, however, the difficulty that was emphasized on page 548: the distribution of the reaction force R on the points of the contact circle becomes statically indeterminate. Since we cannot enter into elastic phenomena, we must make an auxiliary assumption. The most obvious assumption is that the reaction force R is distributed uniformly on the circumference of the contact circle. If we distinguish the points of the circle by an angle β measured about the midpoint of the contact circle, then the fraction $\frac{d\beta}{2\pi}R$ of the entire reaction force R will act on the element $d\beta$ of the circle. This distribution is certainly not entirely correct for a perceptible inclination of the figure axis; it may, however, be permissible for the sake of simplicity.

The following drawing (Fig. 82) refers to the plane of the contact circle. The radius of the contact circle is a, and the radius of the spherical

*) An interesting and technically important consequence of this distinction is given by H. Lorenz in his Lehrbuch der technischen Physik, München 1902, p. 185. The moving control valve of a steam engine may be displaced almost without friction in the direction perpendicular to its motion, in spite of the great steam pressure that bears on it.[212]

end of the figure axis is ϱ. Each of the two component motions is indicated by an arrow. The rotation $\Omega \cos\alpha\, dt$ about the vertical through O is projected in the figure onto the midpoint of the contact circle, and the rotation $\Omega \sin\alpha\, dt$, which it to be imagined about the spherical radius ϱ that lies above the plane of the figure, is projected onto the diameter DD. The azimuth β is also measured from this diameter.

In order to establish the friction force at an arbitrary point P, one must know its motion. The motion of P is composed of the two component motions du and dv; du corresponds to the vertical component of the rotation vector and is directed tangentially to the contact circle, and dv corresponds to the horizontal component of the rotation vector and does not actually lie exactly in the plane of the drawing. The exact direction of dv is given by the common perpendicular to the horizontal component of the rotation vector and the shortest distance from the point in question P to the axis of this component. In so far, however, as the seat is shallow and the radius a is small compared to the radius ϱ, the inclination of dv with respect to the plane of the drawing is small. Thus it is permitted to regard dv as falling in the plane of the drawing. In the same sense, it is allowable to set the distance from the point P to the axis of the horizontal component of the rotation vector, which is actually $b = \sqrt{\varrho^2 - a^2\cos^2\beta}$, simply equal to ϱ. There follow, for the magnitudes of the component motions,

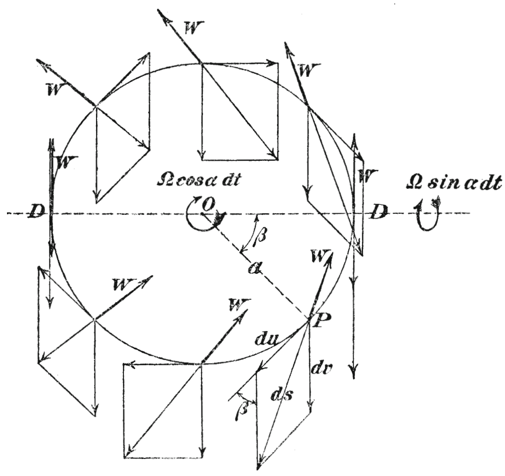

Fig. 82.

$$du = \Omega\cos\alpha\, a\, dt, \quad dv = \Omega\sin\alpha\, b\, dt = \Omega\sin\alpha\, \varrho\, dt.$$

The total motion of P is thus

$$ds = \sqrt{du^2 + dv^2 + 2\, du\, dv \cos\beta}.$$

At each element $d\beta$ of the contact circle, there now appears a friction force W whose direction is opposite to ds and whose magnitude, as a consequence of our assumption on the distribution of the reaction

force, is equal to $\mu R \frac{d\beta}{2\pi}$. The friction force W is constructed in the figure for a number of equidistant points on the circumference of the circle. The work of this friction force is equal to

$$-\mu R \, \frac{d\beta}{2\pi} \, ds,$$

and the total work on the entire contact circle is thus equal to

$$d\mathfrak{A} = -\frac{\mu R}{2\pi} \int_{-\pi}^{+\pi} d\beta \, ds.$$

If we insert the given value of ds, then we can write

$$d\mathfrak{A} = -\frac{\mu R \, \Omega \, dt}{2\pi} \int_{-\pi}^{+\pi} d\beta \, \sqrt{a^2 \cos^2 \alpha + \varrho^2 \sin^2 \alpha + 2a\varrho \cos \alpha \sin \alpha \cos \beta}.$$

This an elliptic integral. Instead of β, we introduce as the integration variable $\gamma = \beta/2$; our integral then takes the form of a Legendre integral of the second kind; it becomes, namely,

$$d\mathfrak{A} = -\mu R \, \Omega \, (a \cos \alpha + \varrho \sin \alpha) \, dt \, \frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} d\gamma \, \sqrt{1 - k^2 \sin^2 \gamma}, \tag{5}$$

where the abbreviation

$$k^2 = \frac{4a\varrho \cos \alpha \sin \alpha}{(a \cos \alpha + \varrho \sin \alpha)^2} = \frac{4 \, \dfrac{\varrho}{a} \, \operatorname{tg} \alpha}{\left(1 + \dfrac{\varrho}{a} \operatorname{tg} \alpha\right)^2} \tag{6}$$

is used. The so-established value (5) of the frictional work differs from the above given value (2), however, only by the factor

$$\frac{1}{\pi} \int_{-\pi/2}^{+\pi/2} d\gamma \, \sqrt{1 - k^2 \sin^2 \gamma} = \frac{2}{\pi} \, E(k), \tag{7}$$

where the designation E is used in the sense of Legendre. To confirm the expression (2), we must therefore ask ourselves to what extent the factor (7) deviates from unity.

To this end, we draw in Fig. 83 both the magnitude of k and that of $\frac{2}{\pi} E(k)$ for changing values of the abscissa $x = \frac{\varrho}{a} \operatorname{tg} \alpha$.

For what concerns the line for k, it is easily shown that this line has a maximum at the abscissa value $x = 1$; the corresponding value

of k is equal to 1. For $x = \frac{1}{2}$ or $x = 2$, $k^2 = \frac{8}{9}$ and $k = 0{,}94$; for $x = \frac{1}{4}$ or $x = 4$, $k^2 = \frac{16}{25}$ and $k = 0{,}80$; for $x = \frac{1}{8}$ or $x = 8$, $k = 0{,}63$, etc.; for $x = 0$ and $x = \infty$, $k = 0$. We thus have a steep ascent at $x = 0$,

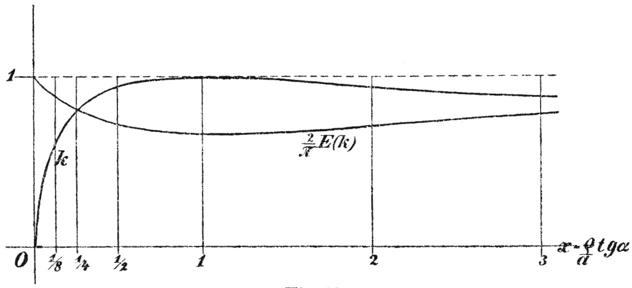

Fig. 83.

followed by a broad maximum and a subsequent asymptotic decrease to zero.

To draw the line for $\frac{2}{\pi}E(k)$, it is enough to use the table of elliptic quadrants included in most tables of logarithms.[213] This table shows, for example, that for the values $k = 0{,}94$, $k = 0{,}80$, and $k = 0{,}63$, $\frac{2}{\pi}E(k)$ is equal to 0,71, 0,81, and 0,89, respectively. For the maximal value $k = 1$, $\frac{2}{\pi}E(k)$ has its smallest value $\frac{2}{\pi} = 0{,}64$, and for $k = 0$ its greatest value 1.

The abscissa value $x = 1$ corresponds to the inclination α between the rotation axis and the vertical for which $\operatorname{tg}\alpha = \frac{a}{\varrho}$, so that the rotation axis passes directly through the circumference of the contact circle. An abscissa value $x < 1$ signifies that the rotation axis passes through the interior of the contact circle, while $x > 1$ means that it falls outside the circle. If, as we assume, the contact circle is small (a small with respect to ϱ), then the rotation axis must be nearly vertical if the rotation axis passes through the circumference of the contact circle or its interior. All reasonably large inclinations of the figure axis correspond in our figure to large values of the abscissa x, and thus to values of $\frac{2}{\pi}E(k)$ near unity. In the opposite case, when the rotation axis passes close to the midpoint of the contact circle, the value of $\frac{2}{\pi}E(k)$ will also be close to 1.

In these two cases, our current expression (5) for the frictional work corresponds approximately with the previous expression (2). *Our previ-*

ous treatment is therefore justified 1) *if the rotation axis forms a perceptible angle with the vertical*, and 2) *if the rotation axis coincides almost entirely with the vertical.* We have spoken previously only of these two cases; the first case (gradual uprighting of the figure axis due to sliding friction) in §§4 and 5, and the second case (gradual extinction of the motion of the upright top as a consequence of boring friction) at the conclusion of §3. If, in contrast, the rotation axis passes through the contact circle or its interior or exterior vicinity, then the previous work expression must be corrected by the inclusion of the factor $\frac{2}{\pi}E(k)$, which in the most unfavorable case ($x = 1$) reduces the work expression to 64% of its previous magnitude.

It is also natural and permissible, from our current standpoint, to decompose the frictional work $d\mathfrak{A}$ into two components $d\mathfrak{A}_1$ and $d\mathfrak{A}_2$, of which one is proportional to the instantaneous rotation angle $\Omega \sin\alpha\, dt = d\omega_h$ about a horizontal axis, and can be called the *work of the sliding friction*, and the other is proportional to the instantaneous rotation angle $\Omega\cos\alpha\, dt = d\omega_v$ about the vertical, and may be called the *work of the boring friction.* With the use of the just-named angles, we can write, according to (5) and (7),

$$d\mathfrak{A} = d\mathfrak{A}_1 + d\mathfrak{A}_2 = -\mu R(\varrho\, d\omega_h + a\, d\omega_v)\,\frac{2}{\pi}\,E(k),$$

and can correspondingly define

$$\text{(8)} \qquad \begin{cases} d\mathfrak{A}_1 = -\mu R\varrho\,\dfrac{2}{\pi}\,E(k)\,d\,\omega_h, \\[2ex] d\mathfrak{A}_2 = -\mu Ra\,\dfrac{2}{\pi}\,E(k)\,d\,\omega_v. \end{cases}$$

These expressions again coincide with the previous values in equation (1) if $\frac{2}{\pi}E(k)$ is approximately equal to 1, and thus if the rotation axis either deviates noticeably from the vertical or almost coincides with it. In the first case, the previous conclusion that the work of the boring friction is small with respect to the work of the sliding friction is valid, and one may therefore disregard the boring friction as an approximation, as we have done in our Omission II. In the other case, in contrast, the work of the boring friction is dominant. If neither of these two cases occurs, then our previous expressions (1) are to be corrected by the inclusion of the factor $\frac{2}{\pi}E(k)$.

Finally, it is completely logical, from our current standpoint, to define the *moments of the sliding friction and the boring friction* in the following manner. One notes, in general, that the moment of a force can be defined as the ratio of the work that the force performs for an infinitesimal rotation about the relevant axis to the magnitude of the rotation angle. In our case we consider, on the one hand, a horizontal axis and the corresponding rotation angle $d\omega_h$. The frictional work in this case is the work of the sliding friction $d\mathfrak{A}_1$. We thus define the quantity

$$M_1 = \frac{d\mathfrak{A}_1}{d\omega_h} = -\mu R\varrho\,\frac{2}{\pi}\,E(k).$$

On the other hand, the work $d\mathfrak{A}_2$ of the boring friction corresponds to the rotation $d\omega_v$ about the vertical. The moment of the boring friction is thus to be denoted as

$$M_2 = \frac{d\mathfrak{A}_2}{d\omega_v} = -\mu Ra\,\frac{2}{\pi}\,E(k).$$

These values naturally coincide once again with the values of M_1 and M_2 given on page 550 of §3 if the rotation axis forms a perceptible angle with respect to the vertical. They show us, moreover, how the previous values are to be corrected if this assumption is not fulfilled. The negative signs that are added in our present definitions of M_1 and M_2 were previously contained in the particular convention that the moments were opposite to the sense of the corresponding rotation components. —

Before we conclude our consideration of friction for the top with a fixed support point, we once again wish to suggest the *experiment* as the appropriate measure of the value of our theoretical results. We have made numerous experiments with the R o z é top illustrated on page 1, and can completely confirm the general outlines of the previously depicted theoretical results. This confirmation, to be sure, is subject to the restriction that the impulse originally imparted to the top be sufficiently large, a restriction, however, that was also laid as the basis of all our theoretical investigations.

For only a moderate impulse, the experimental results are not as typical and clear as for a strong impulse. Perturbing causes that we cannot consider in detail, such as the particular properties of the support

point, then play too great a role compared to the inertial effects that are solely at our command theoretically. The simple schematic description of the influence of friction that is contained in the previous section is not applicable to such cases, and the restricting assumptions also need not apply.

If the impulse is sufficiently strong, however, then the repeatedly named phenomena regularly occur: the figure axis is uprighted as it describes a conical spiral, and indeed does so whether the center of gravity lies above or below the support point; the precessional velocity accelerates, corresponding to equation (2) of page 570; a nutation of the axis that is produced by an intentional blow expires as the figure axis approaches the upright position.

Such a general confirmation of the theory nevertheless leaves still much more to be desired, since it says nothing about the quantitative aspects of the motion. For a thorough experimental confirmation, it would first be necessary to determine the mass and the mass distribution of the top, and therefore to determine the quantities M, P, A, and C by weighing and oscillation experiments, and further to determine the original value and the decrease of the rotational velocity by stroboscopic methods during the individual experiments, and thus indirectly determine the magnitude of the impulse N, and finally to reliably register the changing position of the top.

We are well aware that our treatment is very fragmentary in this direction, and urgently wish that experimental verification and mathematical deliberation be more often treated as equally worthy and equally indispensable factors in future investigations of terrestrial dynamics.

§7. The influence of air resistance on the motion of the top.

In addition to friction, air resistance also acts as an obvious cause of energy dissipation for the motion of the top. Since the top sets the surrounding air into motion, and since this motion is partly communicated to more distant layers of air and partly dissipated by friction between unequally moving layers, kinetic energy continually flows from the moving mass of the top into the surrounding medium. This circumstance cannot help but act in reverse on the motion of the top itself.

The magnitude of the influence will differ according to the form of the top and the type of its motion. With an enlargement of the surface area the influence will generally increase, and with an increase of the mass, for equal surface area, it will decrease. A top of large dimensions will thus be affected less by air resistance than a geometrically similar top of smaller dimensions, since the ratio of spatial volume (or mass) to surface area is larger for the former than for the latter. If the top has vanes against which air is directed, as is the case in pneumatic drives, then the dissipating influence of the air in the subsequent course of the motion will be considerably larger than for a body with a smooth surface.

The type of motion also has an influence on the effect of air resistance. If the surface has rotational symmetry about the figure axis, then only a small amount of air resistance will oppose a simple rotation about the figure axis. The progressive motion of the figure axis, in contrast, will be hindered to a greater degree by air resistance, and a rapid nutation to an again greater degree than a slow precession. One may therefore expect the nutation to decay more rapidly than the slow precessional motion, and the precession again more rapidly than the eigenrotation about the figure axis; air resistance and other energy dissipating effects will always work toward a uniformization of originally present irregularities and a simplification of the form of motion.

As one sees, the problem of air resistance is quite complicated. In order to treat it rigorously, it would be necessary to consider, in addition to the differential equations for the motion of the top, the mutual dependence of the hydrodynamic equations for the motion of the surrounding air, as was already remarked in passing for the analogous ballistic problem (page 535). We would thus encounter a problem that has often been treated from the mathematical side under the name of "the motion of a body in a fluid."*) The underlying assumption of all the relevant treatments is that the fluid is incompressible and frictionless, and that the

*) A comprehensive presentation of the relevant work is given by A. E. H. Love in the Encyklopädie der mathem. Wissensch. Bd. IV, Art. 16, Hydrodynamik II.

body imparts to the fluid a motion with a velocity potential. These assumptions correspond very poorly to the phenomena of air resistance. Without these assumptions, however, there is no possibility of a rigorous and elegant mathematical treatment. We must therefore forgo from the outset an investigation of air resistance in association with the present mathematical literature.

Our treatment should rather be modeled after the manner usually adopted by the physicist for the determination of the pendulum oscillations damped by air resistance, the oscillations of a galvanometer needle with magnetic or fluid damping, etc. One thus assumes that the effect of the damping can be considered with sufficient precision, at least for small amplitudes, if one adds to the equation of motion a term proportional to the instantaneous velocity. We wish to make the analogous assumption *that the effect of air resistance on the motion of the top can be approximately described by a turning-force that is opposite in direction and proportional in magnitude to the instantaneous rotational velocity* Ω. If we resolve Ω into components p, q, r with respect to the three principal axes of the body, then we will set the components of the air resistance with respect to these same axes equal to $-\lambda p$, $-\lambda q$, $-\lambda r$. Perhaps it would be indicated to choose the coefficient of r smaller than that of p and q, and thus to set the components of the moment equal to $-\lambda_1 p$, $-\lambda_1 q$, $-\lambda_2 r$ $(\lambda_2 < \lambda_1)$, since, as noted above, the air will be entrained less by a rotation about the figure axis than by a rotation of the figure axis about an axis perpendicular to it. Since, however, our starting point cannot claim to correspond precisely to the conditions of reality, we will be satisfied with the first-named approximation, which, moreover, has a simple interpretation in the following. Furthermore, we will naturally now disregard all other frictional influences.

We first ask ourselves how the *motion of the force-free top* (the Poinsot motion) is modified by the so-conceived air resistance. The treatment here is very simple. For the Poinsot motion, we had best begin from the Euler equations (cf. page 142), which may be written for the symmetric top $(B = A)$, with the addition of our air resistance terms, as

$$A\frac{dp}{dt} = (A - C)qr - \lambda p,$$

$$A\frac{dq}{dt} = (C - A)rp - \lambda q,$$
$$C\frac{dr}{dt} = \qquad\qquad - \lambda r.$$

One first recognizes from the last equation that the eigenrotation r decreases from its original value r_0 (corresponding to $t = 0$) according to the law

$$r = r_0\, e^{-\frac{\lambda t}{C}}. \tag{1}$$

We combine the two first equations through multiplication by 1 and i into the complex equation

$$A\frac{d\,(p + iq)}{dt} = (C - A)\, ir\,(p + iq) - \lambda(p + iq). \tag{2}$$

Through division by $p + iq$ and insertion of the value of r from (1), there follows

$$\frac{d\,\log\,(p + iq)}{dt} = \frac{C - A}{A}\, ir_0\, e^{-\frac{\lambda t}{C}} - \frac{\lambda}{A},$$

and by integration

$$\log\,(p + iq) = -\frac{\lambda t}{A} - \frac{C - A}{A}\,\frac{C}{\lambda} ir_0\, e^{-\frac{\lambda t}{C}} + \text{const.}$$

If we determine the constant of integration in terms of the initial values p_0, q_0, then we can write

$$p + iq = (p_0 + iq_0)e^{-\frac{\lambda t}{A} + \frac{C-A}{A}\,\frac{C}{\lambda}\, ir_0\left(1 - e^{-\frac{\lambda t}{C}}\right)}. \tag{3}$$

One recognizes here that the absolute value of $p + iq$ (that is, the length of the equatorial component of the rotation vector) decreases according to a similarly simple law as the eigenrotation r. One has, namely,

$$\sqrt{p^2 + q^2} = \sqrt{p_0^2 + q_0^2}\, e^{-\frac{\lambda t}{A}}. \tag{4}$$

If we further denote by α the angle that the equatorial component of the rotation vector encloses with its initial position, in that one sets

$$\frac{p + iq}{\sqrt{p^2 + q^2}} = \frac{p_0 + iq_0}{\sqrt{p_0^2 + q_0^2}}\, e^{i\alpha},$$

then the value of α follows from (3) as

$$\alpha = \frac{C - A}{A}\,\frac{C}{\lambda}\, r_0\left(1 - e^{-\frac{\lambda t}{C}}\right). \tag{5}$$

According to equations (1), (4), and (5), the general character of the motion may now be described in the following manner: *both the com-*

ponent of the rotation vector with respect to the figure axis and the perpendicular component in the equatorial plane are continuously reduced to zero by air resistance; the time duration of this process is infinite; the number of revolutions of the rotation vector about the figure axis in this time interval is finite, and is calculated from (5) *as*

$$\frac{\alpha_\infty}{2\pi} = \frac{C-A}{A}\,\frac{C}{\lambda}\,\frac{r_0}{2\pi};$$

it is always larger as the initial eigenrotation is larger and as the damping constant λ is smaller; with vanishing λ, where the motion becomes a regular precession, this number increases, as it must, to infinity.

An interesting distinction arises according to the relation between the principal moments of inertia A and C. We determine, for example, the instantaneous inclination β of the rotation axis with respect to the figure axis, in that we form, according to (1) and (4),

$$\operatorname{tg}\beta = \frac{\sqrt{p^2+q^2}}{r} = \frac{\sqrt{p_0^2+q_0^2}}{r_0}\,e^{-\lambda t\left(\frac{1}{A}-\frac{1}{C}\right)}.$$

If we further introduce the initial inclination β_0, then we can also write

$$\operatorname{tg}\beta = \operatorname{tg}\beta_0\,e^{-\lambda t\left(\frac{1}{A}-\frac{1}{C}\right)}. \tag{6}$$

The angle β thus constantly increases or constantly decreases, according to whether C is smaller or larger than A. *The rotation axis tends in each case toward the axis of the greatest principal moment of inertia; in the case of the oblate ellipsoid of inertia* ($C > A$) *to the figure axis, and in the case of the prolate ellipsoid of inertia* ($C < A$) *to an equatorial axis.* In the case of the spherical top, where each axis can be conceived as an axis of greatest principal moment of inertia, the position of the rotation axis is naturally not at all changed by air resistance; here, rather, the only effect of air resistance is the gradual weakening of the rotational velocity.

We can describe the distinction between the two cases still more clearly if we imagine the progression of the polhode cone. *In the case $C > A$, the polhode cone becomes narrower in the course of the motion, and is finally contracted to the figure axis after it has encircled the figure axis a finite number of times; in the case $C < A$, the polhode cone broadens, and, once again after a finite number of revolutions, fans out into the equatorial plane.**)

*) If we had made the more general assumption mentioned above that distinguishes between λ_1 and λ_2, then we would have obtained as the conditions for the

In order to illustrate the two cases in a figure, we wish to cut the polhode cone with a plane fixed in the body, which we place at the distance 1 from O, perpendicular to the figure axis. If we call the intersection point of this plane with figure axis O' and the point of intersection with the instantaneous rotation axis P, then the distance $\varrho = O'P$ is equal to $\operatorname{tg}\beta$, and its initial value ϱ_0 is equal to $\operatorname{tg}\beta_0$. The angle through which the vector $O'P$ has turned with respect to its initial position $O'P_0$ is the angle α calculated above. The resulting curve of the successive points P (that is, the trace of the polhode cone in the drawing plane) will then be determined in polar coordinates by ϱ and α. If we introduce as a convenient unit of time the duration τ of a complete rotation of the top at the beginning of the motion, then $r_0 = \dfrac{2\pi}{\tau}$. If we use, moreover, the abbreviations γ and δ for the purely numerical quantities $\dfrac{C}{A} - 1$ and $\dfrac{\lambda\tau}{C}$, then we can write (6) and (5) as

$$\frac{\varrho}{\varrho_0} = e^{-\gamma\delta\frac{t}{\tau}}, \qquad \frac{\alpha}{2\pi} = \frac{\gamma}{\delta}\left(1 - e^{-\delta\frac{t}{\tau}}\right). \tag{7}$$

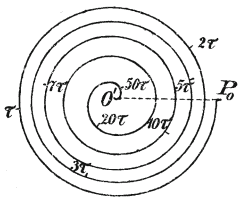

Fig. 84.

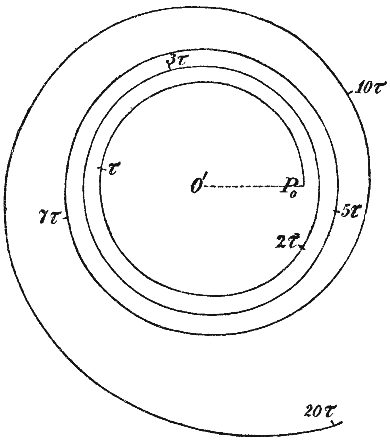

Fig. 85.

In Figs. 84 and 85, we have assumed $\gamma = \pm\frac{1}{2}$, $\delta = \frac{1}{10}$. The number of revolutions of our curve about O' is 5, and the time in which the distance ϱ is decreased to the e^{th} part of its initial value or increased

narrowing or the broadening of the polhode cone the relations

$$\lambda_1 C > \lambda_2 A \quad \text{and} \quad \lambda_1 C < \lambda_2 A.$$

It can thus occur, under some circumstances, that the rotation axis tends to the figure axis even if the figure axis is not an axis of greatest principal moment of inertia.

e-fold is $t = 20\tau$. Both curves can be designated as spirals, but they do not coincide precisely with one of the well-known spiral forms.

We can also account for the antithetical effect of air resistance in the cases $C > A$ and $C < A$ without calculation. We begin from Figs.

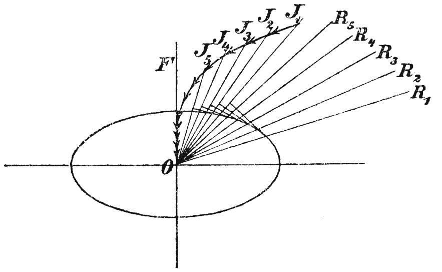

Fig. 86.

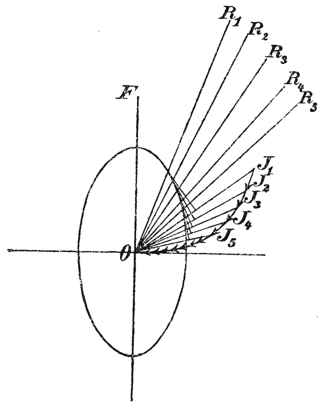

Fig. 87.

86 and 87, which first express, just as Figs. 14 and 13 on pages 107 and 106 do, that the impulse vector lies between the figure axis and the rotation vector in the case $C > A$ of an oblate ellipsoid of inertia, and that the rotation vector is contained between the impulse axis and the figure axis in the opposite case of a prolate ellipsoid of inertia. Now according to our fundamental assumption, the effect of air resistance consists of a moment that is proportional to the rotation vector in magnitude and opposite in direction. We must compose this moment with the instantaneous impulse vector, in that we apply an arrow of length $\lambda\Omega\,\Delta t$ parallel and opposite to the direction of the rotation vector at the endpoint of the impulse vector. *In Fig. 86 the impulse vector will thus approach the figure axis, and in Fig. 87 it will recede from the figure axis.*

The altered position and direction of the impulse also corresponds to a somewhat different position of the rotation axis and a somewhat different magnitude of the rotational velocity. The figure indicates the geometric construction by which, according to the preceding, the direction of the rotation axis can be determined from that of the impulse axis. We must now repeat the above construction, in that we add to the impulse the moment of the air resistance corresponding to the changed position and magnitude of the rotation. As one sees, the impulse and

rotation vectors simultaneously proceed toward the figure axis in Fig. 86, and away from the figure axis in Fig. 87; at the same time, the impulse and the rotation continually decrease in magnitude.

In order to make this process rigorous, one must naturally let the time interval Δt decrease to zero, so that the endpoint of the impulse describes not a broken line path, but rather a continuous curve in the body. Moreover, it would be necessary to consider the change that the impulse may experience in the body due to the "resultant centrifugal turning-force" (cf. page 144). Since, however, this is equal in magnitude and direction to the vector product of the impulse and the rotation vectors, it stands initially perpendicular to the plane of our drawing, and influences neither the magnitude of the impulse nor its inclination with respect to the figure axis. The effect of this turning-force consists, rather, only in turning the corresponding positions of the impulse and rotation axes out of the drawing plane by an angle that increases with the progressing motion, in such a way that the endpoint of the impulse describes not a plane curve, but rather a spiral curve that winds around the figure axis. The precise form of the curve, moreover, is contained in the above calculation, since indeed the coordinates of the endpoint of the impulse relative to the body differ from the components p, q, r of the rotation vector only by the factors A and C.

But the form of the curve described by the endpoint of the impulse *in space* is also essentially clear from the preceding. With respect to space, the endpoint of the impulse is instantaneously displaced in a direction parallel and opposite to the rotation axis. According to the previous figures, the rotation axis continually approaches the impulse axis, and, moreover, turns about its instantaneous position, since the figure axis turns about the instantaneous position of the rotation axis and the position of the rotation axis is also determined by the positions of the figure and impulse axes. One thus concludes that the endpoint of the impulse vector must describe a screw-shaped line with decreasing winding breadth about a certain mean direction, as we have indicated schematically in the adjacent figure.*) The impulse vector therefore does not remain precisely constant in space as it does for the ideal Poinsot motion, but its mean direction remains constant, and the oscillations about the mean position decrease in the course of the motion.

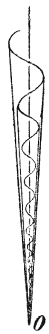
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Fig. 88.

*) The spiral line naturally goes alternately behind and in front of the mean line, which is not expressed with sufficient clarity in the figure.

The motion of the figure axis in space again proves to be fundamentally different in the two cases $C > A$ and $C < A$. In the former case, the figure axis tends to a direction that ultimately coincides with the direction of the rotation axis, and thus also with the direction of the impulse axis. In the latter case, the figure axis ultimately stands perpendicular to the direction of the impulse and the rotation axes. If we designate, for example, the mean direction of the impulse axis as the vertical, then we can say that *the figure axis is uprighted by air resistance for the oblate top, and is lowered for the prolate top.* After the motion has expired (that is, after an infinitely long time), the figure axis stands vertically in the first case, and horizontally in the second case.[214]

With consideration of air resistance, our previous stability distinction for the symmetric top (cf. pages 132 and 133) is subjected to a fundamental revision. We said previously that *uniform rotation about the figure axis is a stable form of motion, and that uniform rotation about an equatorial axis is an unstable form of motion.* Each statement is only half correct if we consider the effect of air resistance. We impart to the top a rotation exactly about the figure axis. This is, with consideration of air resistance, a possible permanent form of motion, in so far as the rotation axis remains unchanged in the body and in space, and the rotational velocity only decreases gradually. If the initial rotation, however, is not exactly about the figure axis, or is deflected somewhat by an additional impulse, then the oblate top behaves oppositely to the prolate. For the oblate top, the rotation axis strives, due to air resistance, to unite itself with the figure axis, and soon stands perceptibly still in space. For the prolate top, an arbitrarily small initial deviation of the rotation axis from the figure axis becomes larger and larger, and likewise for the impulse axis. Or, otherwise expressed, the figure axis that initially coincides perceptibly with the rotation axis and the impulse axis finally becomes, in the course of the motion, perpendicular to these axes. We thus recognize that *the rotation about the figure axis is, with consideration of air resistance, stable for the oblate top, and labile for the prolate top.* The opposite holds for a rotation about an equatorial axis.

The preceding remarks partly coincide, with respect to their analytic character, with the deliberations of S t o n e*) on the developmental

*) On the possibility of a change in the position of the Earth's axis due to a frictional action associated with the phenomena of the tides. Monthly notices of the astronomical Society, London, March 1867.[215]

history of the Earth. From the geological side, the hypothesis has often been made that the rotation axis of the Earth may have had a different location in the body of the Earth in earlier geological periods. Can this assumption be reconciled, on the basis of a frictional resistance (tidal friction) of the type presumed here, with the fact that the rotation axis now coincides almost precisely with the polar axis? Since the Earth is an oblate symmetric top and its polar axis is the axis of the greatest principal moment of inertia, it would, according to the preceding, be possible. We will be led in the next chapter, however, with consideration of the numerous circumstances of the process, to a negative answer to the proposed question. —

We now supplement the above deliberations by the consideration of gravity. We wish in the following, however, to disregard the inequality of the moments of inertia, and thus consider a heavy *spherical* top. While the rotation of a force-free spherical top with air resistance always occurs about an axis fixed in the body and in space, the result for the heavy spherical top must be that the center of gravity axis through O ultimately points vertically downward in all cases. We must demonstrate the process by which this is achieved in an approximate manner.

For the heavy top it is convenient, as has often been remarked, to use the Euler angles and the Lagrange equations. We first determine the components (or moments) of the air resistance with respect to the three coordinates φ, ψ, ϑ. They may be denoted by $\mathsf{\Phi}$, $\mathsf{\Psi}$, $\mathsf{\Theta}$, and are proportional to the perpendicular projections of the rotation vector (or the linear path consisting of the segments φ', ψ', ϑ' illustrated in Fig. 76) onto the figure axis, the vertical, and the line of nodes. If, as previously, one calls the factor of proportionality λ, then one finds, according to Fig. 76,

$$(8) \qquad \mathsf{\Phi} = -\lambda(\varphi' + \psi'\cos\vartheta), \quad \mathsf{\Psi} = -\lambda(\psi' + \varphi'\cos\vartheta), \quad \mathsf{\Theta} = -\lambda\vartheta'.$$

The corresponding impulse components are called N, n, and $[\mathsf{\Theta}]$; as follows, for example, from equations (2) of the fourth section, one has, in the case of the spherical top,

$$(9) \qquad N = A(\varphi' + \psi'\cos\vartheta), \quad n = A(\psi' + \varphi'\cos\vartheta), \quad [\mathsf{\Theta}] = \mathsf{A}\vartheta'.$$

The Lagrange equations are, if T signifies the *vis viva* of the spherical top,

$$\frac{dN}{dt} = \mathsf{\Phi}, \quad \frac{dn}{dt} = \mathsf{\Psi}, \quad \frac{d[\mathsf{\Theta}]}{dt} - \frac{\partial T}{\partial\vartheta} = P\sin\vartheta + \mathsf{\Theta}.$$

With consideration of (8) and (9) and the value of $\partial T/\partial\vartheta$ given in equation (4) of the fourth section, we can write

$$\frac{dN}{dt} = -\frac{\lambda}{A}N, \quad \frac{dn}{dt} = -\frac{\lambda}{A}n, \tag{10}$$

$$A\vartheta'' + \lambda\vartheta' = P\sin\vartheta - \frac{(n - N\cos\vartheta)(N - n\cos\vartheta)}{A\sin^3\vartheta}. \tag{11}$$

Equations (10) determine, as for the force-free top, an exponential decrease of the impulse components according to the laws

$$N = N_0 e^{-\frac{\lambda t}{A}}, \quad n = n_0 e^{-\frac{\lambda t}{A}}; \tag{12}$$

the ratio of the two components thus remains constant, since $N:n = N_0:n_0$. If we insert the values (12), then (11) simplifies to

$$A\vartheta'' + \lambda\vartheta' = P\sin\vartheta - e^{-\frac{2\lambda t}{A}}\,\frac{(n_0 - N_0\cos\vartheta)(N_0 - n_0\cos\vartheta)}{A\sin^3\vartheta}. \tag{13}$$

We begin here, as for the previous friction problem, with "precession-like motions." Since for regular precession $\vartheta =$ const., we now wish to ask for motions in which ϑ' and ϑ'' are small. In the first approximation, we thus set the left-hand side of (13) to zero and follow a procedure that finds, as the "method of slow motions," extraordinarily many conscious or unconscious applications in all fields. The sense of the procedure is to conceive of a sufficiently slow motion in an approximate manner as a series of equilibrium positions, in that one disregards the inertia of the system, which is obviously always less important as the motion is slower. This procedure provides, in many cases, a useful first approximation to the actual course of the motion, an approximation that we will confirm, in the present case, by the calculation of a second approximation.

We therefore determine $\cos\vartheta$ as a function of t by the equation

$$\left(\frac{n_0}{N_0} - \cos\vartheta\right)\left(\frac{N_0}{n_0} - \cos\vartheta\right) = \frac{AP}{n_0 N_0}\sin^4\vartheta\, e^{\frac{2\lambda t}{A}}. \tag{14}$$

We assume a strong initial impulse, and therefore assume that AP/N_0^2 is small; AP/n_0N_0 is of the same order of magnitude. If the initial impulse falls, moreover, in the approximate direction of the figure axis, then n_0/N_0 will be a proper fraction, which, for example, can be set equal to the cosine of an auxiliary angle ϑ_0. We distinguish the beginning of the motion (t small) and the end of the motion (t very large).

a) *t small.* The right-hand side of (14) is, because of AP/n_0N_0, small; on the left-hand side, one of the two factors must likewise be

small. This can only be the factor $\frac{n_0}{N_0} - \cos\vartheta = \cos\vartheta_0 - \cos\vartheta$. We correspondingly set $\cos\vartheta = \cos\vartheta_0 + \varepsilon$, and neglect the higher powers of ε. From (14) there follows

$$-\varepsilon\Big(\frac{1}{\cos\vartheta_0} - \cos\vartheta_0\Big) = \frac{AP}{n_0 N_0}\sin^4\vartheta_0\, e^{\frac{2\lambda t}{A}},$$

$$\varepsilon = -\frac{AP}{N_0^2}\sin^2\vartheta_0\, e^{\frac{2\lambda t}{A}},$$

and

$$\cos\vartheta = \cos\vartheta_0 - \frac{AP}{N_0^2}\sin^2\vartheta_0\, e^{\frac{2\lambda t}{A}}; \tag{15}$$

with the same degree of approximation,

$$\vartheta = \vartheta_0 + \frac{AP}{N_0^2}\sin\vartheta_0\, e^{\frac{2\lambda t}{A}}. \tag{15'}$$

We thus conclude that at the beginning of the motion ϑ increases, and the figure axis therefore descends if P is positive; that is, if the center of gravity lies on the positive figure axis. In the opposite case, ϑ decreases and the figure axis rises, while the center of gravity axis, which is identical with the negative figure axis, descends. The initial value of ϑ coincides approximately with our auxiliary angle ϑ_0.

b) t *large.* The product $\frac{AP}{n_0 N_0}e^{\frac{2\lambda t}{A}}$ becomes arbitrarily large if t grows without bound. Since the left-hand side of (14) remains finite, $\sin\vartheta$ must become small with increasing t; $\cos\vartheta$ must therefore become equal to ± 1. For $\cos\vartheta = +1$, the left-hand side of (14) is equal to

$$-\frac{(N_0 - n_0)^2}{n_0 N_0},$$

and for $\cos\vartheta = -1$ it is equal to

$$+\frac{(N_0 + n_0)^2}{n_0 N_0}.$$

If one compares the signs on the right- and left-hand sides of (14), one recognizes that $\cos\vartheta = +1$ in the case $P < 0$ and $\cos\vartheta = -1$ in the case $P > 0$. In both cases, the center of gravity axis is directed downward at the end of the motion. The formulaic representation of ϑ at the end of the motion is thus

$$\begin{cases} P > 0 \quad \sin\vartheta = \pi - \vartheta = \sqrt[2]{\dfrac{N_0 + n_0}{\sqrt{AP}}}\, e^{-\frac{\lambda t}{2A}}, \\ P < 0 \quad \sin\vartheta \quad = \vartheta = \sqrt[2]{\dfrac{N_0 - n_0}{\sqrt{-AP}}}\, e^{-\frac{\lambda t}{2A}}. \end{cases} \tag{16}$$

The graphical course of the dependence between ϑ and t determined by (14) is illustrated schematically by the two curves of Fig. 89.

It is naturally by no means certain that our formulas, found by the rather arbitrary neglect of some terms in the differential equation (13), approximate the actual course of the precession-like motion. In any case, a proof that the neglected terms are actually small compared to the retained terms is still necessary. In that we provide this verification after the fact, we will show, at the same time, the possibility of improving our present approximation stepwise.

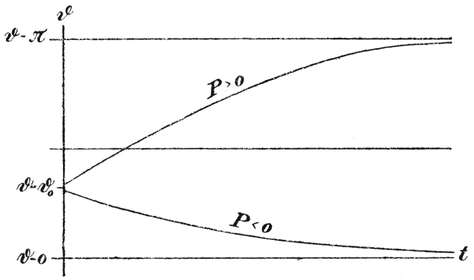

Fig. 89.

a) t *small.* We calculate the left-hand side of equation (13) on the basis of formula (15′). There follows

$$A\vartheta'' + \lambda\vartheta' = \frac{6\lambda^2}{A} \cdot \frac{AP}{N_0^2} \sin\vartheta_0\, e^{\frac{2\lambda t}{A}}. \tag{17}$$

The ratio of this expression to the first term of the right-hand side of (13), which we can write as approximately equal to $P\sin\vartheta_0$, is

$$6\,\frac{\lambda^2}{N_0^2}\, e^{\frac{2\lambda t}{A}}.$$

The order of magnitude of this ratio is, since for small t the exponential quantity has only a moderate value, determined by the number λ^2/N_0^2. As a consequence of the introduction of the factor of proportionality λ, however, $2\pi\lambda$ signifies the additional impulse that the air resistance exerts for a full rotation about an arbitrary axis. We may assume that this is considerably smaller than the eigenimpulse of the top, or, otherwise expressed, that the inertial effect of the air is extremely small compared to the inertial effect of the top. Our confirmation of the above approximate solution therefore gives a satisfying result, in that it shows that the neglected terms of the differential equation are, in fact, of a smaller order than the retained terms.

The solution thus far may now be easily corrected by consideration of the left-hand side of (13). We set everywhere in (13), except in the

term $n_0 - N_0 \cos\vartheta$ that gives the result, our first approximation (15). For the left-hand side we use the expression (17); on the right-hand side we must set, according to (15),

$$\sin\vartheta = \sin\vartheta_0\Big(1 + \frac{AP}{N_0^2}\cos\vartheta_0\, e^{\frac{2\lambda t}{A}}\Big),$$

$$N_0 - n_0\cos\vartheta = N_0^2\sin^2\vartheta_0\Big(1 + \frac{AP}{N_0^2}\cos\vartheta_0\, e^{\frac{2\lambda t}{A}}\Big).$$

After a few formal simplifications and the neglect of the higher powers of the small terms, we find

$$n_0 - N_0\cos\vartheta = \frac{AP}{N_0}\sin^2\vartheta_0\, e^{\frac{2\lambda t}{A}}\left\{1 + \left(\frac{3AP}{N_0^2}\cos\vartheta_0 - \frac{6\lambda^2}{N_0^2}\right)e^{\frac{2\lambda t}{A}}\right\},$$

and thus

$$\text{(18)}\quad \cos\vartheta = \cos\vartheta_0 - \frac{AP}{N_0^2}\sin^2\vartheta_0\, e^{\frac{2\lambda t}{A}}\left\{1 + \left(\frac{3AP}{N_0^2}\cos\vartheta_0 - \frac{6\lambda^2}{N_0^2}\right)e^{\frac{2\lambda t}{A}}\right\}.$$

This is the desired correction of (15), which we must regard as a second approximation, since the squares and products of the small quantities AP/N_0^2 and λ^2/N_0^2 are retained. Evidently, our procedure contains the germ of a power series expansion of arbitrarily high order in just these small quantities; to achieve this, we need only calculate a third approximation from the second, etc. The convergence of the expansion would become poor with increasing t, since the powers of the named small quantities in (18) would be accompanied by the corresponding powers of the factor $e^{2\lambda t/A}$. For this reason, and because of the detailed nature of the resulting formulas, we will be satisfied with the second approximation.

b) t *large.* Here too it is first necessary to verify that the left-hand side of equation (13) may be neglected for the construction of the approximate solution (16). We calculate, for this purpose, the neglected terms according to equation (16), and find, according to whether $P \gtrless 0$,

$$A\vartheta'' + \lambda\vartheta' = \pm\frac{1}{4}\frac{\lambda^2}{A}\sin\vartheta.$$

The ratio of this expression to the first term $P\sin\vartheta$ of the right-hand side of (13) is

$$\pm\frac{1}{4}\frac{\lambda^2}{AP}.$$

We may assume that λ^2 is small compared with the equidimensional quantity AP, which corresponds to the assumption that the influence of air resistance on the motion of the top is small compared to the effect of

gravity. In any case, the neglect of the left-hand side of (13) is justified under this assumption in the first approximation.

If we continue here too along this path to a second approximation, then we find, if we distinguish the cases $P \gtrless 0$ by a double sign,

$$\sin\vartheta = \sqrt{\frac{N_0 \pm n_0}{\sqrt{\pm AP}}}\, e^{-\frac{\lambda t}{2A}} \left(1 \pm \frac{1}{16}\frac{\lambda^2}{AP}\right), \tag{19}$$

and therefore obtain a formula of the same character as (16). One can also proceed here to a third approximation, etc.

Figure 89, created for the schematic representation of the first approximation, can serve just as well for the illustration of this second approximation. Moreover, this figure can be overlaid with a nutation of the period of the rotation of the top, which depends on the initial conditions of the motion and is smoothed in the course of the motion. Our curve in Fig. 89 can be employed only for appropriately chosen initial conditions (and then only approximately), as in the case of regular precession for the ideal top. In general, it will represent not the integral curve itself, but rather only the "guideline" of the integral curve, about which the latter oscillates with decreasing amplitude, as in Figs. 79 and 80 of the fourth section.

For a nonspherical ellipsoid of inertia, the phenomena are essentially more complicated. It can occur, in this case, that the figure axis will tend toward the vertical due to the combined action of gravity and air resistance, but be deflected from the vertical due to the differences of the principal moments of inertia. Which of these influences will obtain the upper hand cannot be decided without a deeper entry.[216]

§8. The elasticity of the material of the top.

As indispensable as the concept of the rigid body is for natural science and technology, it is certain that it is, in reality, only a rough approximation. The spinning top will not move altogether as a rigid body, but will simultaneously be somewhat deformed by the stresses that are generated by the motion. The question is only whether such changes of form will be perceptible under any given circumstances. This question has become of direct importance in the case for which we may perhaps be most inclined to adopt the representation of a rigid

constitution, the case of our Earth. Not only is the form of the Earth influenced in a continuous manner by its rotation and is different from what it would be if the Earth one day ceased to rotate, but the form of the Earth also changes if the position of the rotation axis changes in the body of the Earth, and thus is displaced somewhat from its normal or mean position in which it coincides with the figure axis of the Earth. Such a change of form is naturally not measurable. A change of form, however, exerts a reverse effect on the motion of the Earth, an effect on the change of the rotation axis in the body of the Earth that is very well possible to measure. We will return to this phenomenon in the next chapter. Here it is necessary to prepare the later discussion, and to show that the corresponding questions occur for any type of top problem, even if they are hardly of noteworthy significance for the usual dimensions and form of our apparatus.

Keeping in mind the proportions of the Earth, we consider a top *with the form of an oblate ellipsoid of revolution.* The mass distribution in the interior of the ellipsoid is assumed to be homogeneous. We imagine that the center of gravity itself is supported, so that we have to consider only force-free motion, which, as we know, is regular precession for the rigid symmetric top. We will direct our interest to the period of the precession and the influence of the material elasticity on this period.

With respect to terminology, it is to be kept in mind that the motion to be studied here is designated in the case of the Earth as *free nutation* (and in particular, in so far as one disregards the influence of elasticity, as *Euler nutation*), while one usually understands by the precession of the Earth a motion with a much longer period that is *forced* by the attraction of the Sun and the Moon. This slow forced precession is overlaid on the more rapid Euler nutation, our force-free precession, so that the total motion takes on the well-known character of a pseudoregular precession. Moreover, we also designated the oscillation of the top with respect to the motion of forced regular precession as a nutation in the general investigation of pseudoregular precession in Chap. V, §2; there too this nutation proved to be equivalent to force-free precession of the top without gravity under the conditions by which we defined pseudoregular precession; namely, that 1) the eigenimpulse was very large (N^2 large compared with AP)

and 2) the figure axis always lay in the vicinity of the impulse axis, conditions that are fulfilled in the case of the Earth.

The form and mass-distributional properties of the top will be specified by giving the assumed positive ratio

$$\varepsilon = \frac{C - A}{A},$$

which we call the "*ellipticity.*" The numerical eccentricity e of the meridian curve is calculated from the ellipticity according to the formula $e = \sqrt{2\varepsilon/(1+\varepsilon)}$. For an arbitrary ellipsoid, the moment of inertia about an arbitrary principal axis is equal to the fifth part of the mass multiplied by the sum of the squares of the other two principal axes. If we denote by b the minor principal axis of the ellipsoid that coincides with the figure axis and by a the major principal axis in the equatorial plane, then one has

$$A = \frac{M}{5}(a^2 + b^2), \quad C = \frac{M}{5}(a^2 + a^2),$$

and thus

$$\varepsilon = \frac{a^2 - b^2}{a^2 + b^2},$$

while the well-known definition of the numerical eccentricity is

$$e^2 = \frac{a^2 - b^2}{a^2}.$$

The relationship given above between ε and e thus follows easily.

In that we once again consider the proportions of the Earth, *we assume ε to be a small number*; the form of the ellipsoid then deviates little from a spherical form (spheroid). Under this assumption, we write the approximate equation of the surface of the ellipsoid. If z is measured in the direction of the figure axis and x and y are measured with respect to two rectangular axes in the equatorial plane, then we first have, without approximation,

$$\frac{z^2}{b^2} + \frac{x^2 + y^2}{a^2} = 1.$$

We transform this equation into central polar coordinates, in that we designate r as the distance of a point of the surface from the center of the ellipsoid, and Θ as the inclination of the radius vector r with respect to the equatorial plane. We then have

$$z^2 = r^2 \sin^2 \Theta, \quad x^2 + y^2 = r^2 \cos^2 \Theta,$$

and therefore, as a result of the equation of the ellipsoid above,

$$\frac{1}{r^2} = \frac{1 - \cos^2 \Theta}{b^2} + \frac{\cos^2 \Theta}{a^2} = \frac{1}{b^2}\,(1 - e^2 \cos^2 \Theta),$$

and, approximately,

$$r = b(1 + \varepsilon \cos^2 \Theta). \tag{1}$$

This is the original form of the top. If we now set the top into rotation, a change of shape occurs, which we will assume to be small. If the rotation occurs directly about the figure axis, then the ellipsoid will become somewhat more oblate; the ellipticity ε will be increased by a small amount ε'. For the calculation of the additional ellipticity ε', which must follow from the fundamental principles of the theory of elasticity, one may unquestionably disregard the originally present ellipticity ε, and thus assume the original form of the top to be a simple sphere. For the small deviation ε from spherical form will influence the additional ellipticity ε' only by a quantity of the second order (the order of the product $\varepsilon\varepsilon'$). One can thus pose the question of what radius one should give the sphere by which one will replace, for the purpose of the calculation of ε', the original ellipsoid with principal axes a and b. It is most natural to choose the radius as a mean length m between a and b, which one determines so that the volume of the sphere will be equal to the volume of the original ellipsoid. This requirement leads to the condition

$$m^3 = a^2 b.$$

If one inserts for a the value $a = b(1+\varepsilon)$ that follows from equation (1) with $\Theta = 0$, then

$$m^3 = b^3(1 + 2\varepsilon), \quad m = b\Big(1 + \frac{2}{3}\,\varepsilon\Big), \quad b = m\Big(1 - \frac{2}{3}\,\varepsilon\Big).$$

The equation of the original ellipsoid may thus be written as

$$r = m\Big(1 + \varepsilon(\cos^2 \Theta - \frac{2}{3})\Big), \tag{2}$$

while the equation of the ellipse into which the sphere of radius m is transformed will be

$$r = m\Big(1 + \varepsilon'(\cos^2 \Theta - \frac{2}{3})\Big). \tag{2$'$}$$

By superposition of the two small oblatenesses ε and ε', the equation of our ellipsoid that is deformed by the rotation follows as

$$r = m\Big(1 + (\varepsilon + \varepsilon')(\cos^2 \Theta - \frac{2}{3})\Big). \tag{3}$$

We can proceed similarly if the rotation occurs about an axis different from the figure axis. Let the angle between the figure axis and the rotation axis be δ (cf. Fig. 90, $\angle FOR$). The oblateness that now results

will be distributed symmetrically about the rotation axis, and can again be calculated as if the top had the original form of a sphere of radius m. The additional ellipticity ε' has the same magnitude as previously. The equation of the ellipsoid that results from the sphere is now

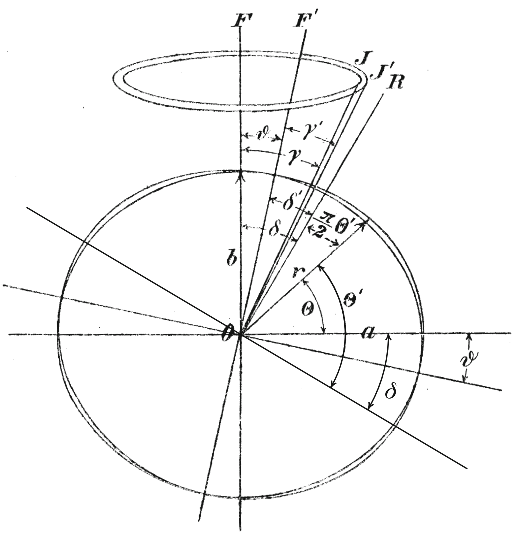

Fig. 90.

$$(4)\quad r = m\Big(1+\varepsilon'(\cos^2\Theta' - \frac{2}{3})\Big),$$

where Θ' signifies the angle of the arbitrary radius vector r with respect to the plane perpendicular to the rotation axis. As one recognizes from Fig. 90,[217]

$$\Theta' = \Theta + \delta.$$

Thus the change of form that the original ellipsoid (equation (2)) suffers in the rotation is also determined with sufficient accuracy. The new form will be given by the equation

$$(5)\qquad r = m\Big(1 + \varepsilon\cos^2\Theta + \varepsilon'\cos^2(\Theta+\delta) - \frac{2}{3}(\varepsilon+\varepsilon')\Big).$$

This is, with the same approximation, the equation of an oblate ellipsoid, as were equations (1) to (4) above. The figure axis of the new ellipsoid no longer coincides, however, with the original figure axis.

For the determination of the new figure axis (and the corresponding equatorial plane), we have the equation

$$\frac{dr}{d\Theta} = 0;$$

that is,

$$\varepsilon\cos\Theta\sin\Theta + \varepsilon'\cos(\Theta+\delta)\sin(\Theta+\delta) = 0.$$

We will assume the angle δ to be *small.* We may thus write, instead of the equation above,

$$\varepsilon\cos\Theta\sin\Theta + \varepsilon'\cos\Theta\sin\Theta + \varepsilon'\delta(\cos^2\Theta - \sin^2\Theta) = 0,$$

or

$$\operatorname{tg} 2\Theta = -\frac{2\varepsilon'\delta}{\varepsilon+\varepsilon'}.$$

The right-hand side is small because of the factor δ; one thus obtains for Θ two values $-\vartheta$ and $\pi/2-\vartheta$ that differ little from 0 and $\pi/2$. The former pertains to the principal axis lying in the equatorial plane, and the latter to the new figure axis. The angle between the original and the new figure axis ($\angle FOF'$ in Fig. 90) is likewise ϑ, and one has, with sufficient accuracy,

$$\vartheta = \frac{\varepsilon'}{\varepsilon+\varepsilon'}\,\delta < \delta. \tag{6}$$

The result thus obtained is very intuitive:

Were the material of the top absolutely rigid, the mass distribution would remain symmetric, after as before, about OF ($\vartheta = 0$); were it absolutely compliant (fluid), then it would be arranged symmetrically about the rotation axis OR ($\vartheta = \delta$); for each finite degree of elastic resistance, it must form an intermediate state, in which an axis of symmetry of the mass distribution will lie between OF and OR ($\vartheta < \delta$).

From the now known form of the top, it will be easy to determine, always under the assumption of a homogeneous mass distribution, the moments of inertia and the ellipticity of the deformed spheroid. And indeed, we will be led to the same ellipticity whether the rotation occurs about the figure axis OF or about the differing axis OR, and whether the surface is given by equation (3) or by equation (5).

In fact, we obtain as the principal axes of the deformed ellipsoid

$$\begin{array}{llll}
\text{from eqn. (3)} & \text{for} & \Theta = 0 & : \; r = a' = m\Big(1+\frac{1}{3}(\varepsilon+\varepsilon')\Big),\\
\quad ,, & \quad ,, & \Theta = \frac{\pi}{2} & : \; r = b' = m\Big(1-\frac{2}{3}(\varepsilon+\varepsilon')\Big),\\
\text{from eqn. (5)} & \text{for} & \Theta = -\vartheta & : \; r = a' = m\Big(1+\frac{1}{3}(\varepsilon+\varepsilon')\Big),\\
\quad ,, & \quad ,, & \Theta = \frac{\pi}{2}-\vartheta & : \; r = b' = m\Big(1-\frac{2}{3}(\varepsilon+\varepsilon')\Big),
\end{array}$$

where, because of the smallness of ϑ, $\cos^2\vartheta = 1$ and $\sin^2\vartheta = 0$ were assumed. *The two ellipsoids are therefore, in the first approximation, congruent; they differ only in position, and not in form.* Correspondingly, their principal moments of inertia A' and C' and their ellipticity E will also be equal. One finds immediately, from the preceding values of a' and b',

$$\left\{\begin{aligned}
A' &= \frac{M}{5}(a'^2+b'^2) = \frac{2M}{5}\Big(1-\frac{1}{3}(\varepsilon+\varepsilon')\varepsilon\Big),\\
C' &= \frac{M}{5}(a'^2+a'^2) = \frac{2M}{5}\Big(1-\frac{1}{3}(\varepsilon+\varepsilon')\Big),\\
\mathsf{E} &= \frac{C'-A'}{A'} = \varepsilon+\varepsilon'.
\end{aligned}\right. \tag{7}$$

We are now in a position to calculate the *period of the free precession of the top*, both for a rigid material and for a material that is deformed by the rotation. In the first case, we can call upon the calculation of Chapter III, §2. According to equation ($6'$) of page 151, the equatorial component $p + iq$ of the rotation vector is given for regular precession by an exponential expression, in which the exponent it is multiplied by the factor

$$\frac{C - A}{A} r_0.$$

This factor must therefore be equal to $2\pi/T$, where T signifies the precession period. If, on the other hand, τ signifies the period of one rotation of the top, then the rotation vector Ω is equal to $2\pi/\tau$, and its component with respect to the figure axis, which in the cited equation is denoted by r_0, is equal to $\cos\delta \cdot 2\pi/\tau$, understanding by δ the angle drawn in Fig. 90. One thus has

$$\frac{2\pi}{T} = \frac{C - A}{A} \frac{2\pi}{\tau} \cos\delta,$$

or, since one may set $\cos\delta$ equal to 1 with sufficient accuracy,

$$T = \frac{\tau}{\varepsilon}. \tag{8}$$

More instructive and more useful for the sequel, however, is the following path to the derivation of this formula. According to our conception, the Euler equations state that the magnitude and direction of the impulse vector remain unchanged in space for force-free motion, and that, in contrast, the progressional velocity of the impulse vector with respect to the top is equal, in magnitude and direction, to the vector product of the impulse and the rotation vectors (the so-called "resultant centrifugal turning-force"). If J signifies the impulse vector, $|J|$ its length, dJ its instantaneous change relative to the top, and J forms the angle γ with the figure axis (cf. Fig. 90), then one has

$$\frac{dJ}{dt} = V(J, R) = |J|\,\Omega \sin(\delta - \gamma). \tag{9}$$

The quantity Ω is the length of the rotation vector R, and can be set equal, as above, to $2\pi/\tau$. During a precessional circuit, the endpoint of the impulse now describes in the top a circle of radius $|J|\sin\gamma$ about the figure axis. Thus it takes, due to the given value of its progressional velocity, the time

$$T = \frac{2\pi |J| \sin\gamma}{|J|\Omega \sin(\delta - \gamma)} = \tau \frac{\sin\gamma}{\sin(\delta - \gamma)},$$

or, for sufficient smallness of the angle δ,

$$T = \tau \frac{\gamma}{\delta - \gamma}. \tag{8'}$$

This result naturally coincides with that given in (8). The quantity $\operatorname{tg}\gamma$, namely, is equal to the ratio of the equatorial component of the impulse to its component with respect to the figure axis, and $\operatorname{tg}\delta$ is equal to the ratio of the corresponding components of the rotation vector. Since, according to the fundamental relation between the impulse vector and the rotation vector, the corresponding components of the two vectors are related to each other as A is to C, respectively, there follows

$$\operatorname{tg}\gamma : \operatorname{tg}\delta = A : C,$$

or, with sufficient accuracy,

$$\gamma : \delta = A : C; \qquad \frac{\gamma}{\delta - \gamma} = \frac{A}{C - A} = \frac{1}{\varepsilon}. \tag{10}$$

The deliberation that leads to equation (9) may be carried over immediately to the elastically deformable top. We must only emphasize one assumption explicitly: *the change of form should have time to be completely developed in the manner described above, for each position of the rotation vector, before the position of this vector in the top has changed perceptibly.* This assumption is justified to a high degree, since, generally speaking, stresses and form changes propagate with the relevant speed of sound of the material, while the observable motion phenomena (here the change of position of the rotation vector) follow much more slowly. We will see, under this assumption, that the general motion of the deformable top can also be designated as regular precession. Were, in contrast, this assumption not permissible, then the position and magnitude of the oblateness would lag behind the oblateness determined by the instantaneous position of the rotation vector, and the motion would be much more complicated.

We distinguish the *instantaneous figure axis* OF' in Fig. 90 (the symmetry line of the mass distribution that is changed by the instantaneous rotation) from the *original* or the *mean figure axis* OF. The motion of the deformable top will now be, at each instant, the same as that of a rigid top with the changing figure axis OF' and the changed moments of inertia A' and C'. Correspondingly, the endpoint of the impulse, whose progressional velocity with respect to the material of the top is again given by the vector product of the impulse vector (J') and the rotation vector (R), will progress at each instant in a direction perpendicular to the plane that lies through the vectors J' and R. Each change of

position of J' brings with it, however, a change of R, and indeed the endpoint of R must progress, according to the general relation between the impulse vector and the rotation vector, parallel to the endpoint of the impulse. Each change of position of the rotation vector has as a result, on the other hand, a change of form and a new position of the instantaneous figure axis. Since we assume that the development of the form change is fully completed, the instantaneous figure axis always lies in the plane determined by OF and R. Because of the general relation between the impulse vector and the rotation vector, the axis of J' also lies in this plane. The three axes OF', OJ', and OR therefore lie in the same meridian through OF, which turns about OF. Since, moreover, the angular distances between the three axes are constant, each of the axes describes a circular cone, and, in particular, the impulse endpoint J' describes a circle about OF. This motion has the complete character of a regular precession; in addition to the impulse and rotation axes, however, the instantaneous figure axis also progresses in the body. We now calculate the precession period T', in that we again divide the path length of one circuit of the impulse endpoint by its progressional velocity.

The length of the path is $2\pi|J'|\sin(\vartheta+\gamma')$ (cf. Fig. 90), and the progressional velocity is

$$\frac{dJ'}{dt} = |J'|\,\Omega\sin(\delta'-\gamma');$$

the period of the precession is thus

$$T' = \frac{2\pi|J'|\sin(\vartheta+\gamma')}{|J'|\Omega\sin(\delta'-\gamma')} = \tau\,\frac{\sin(\vartheta+\gamma')}{\sin(\delta'-\gamma')},$$

or, with sufficient accuracy,

$$T' = \tau\,\frac{\vartheta+\gamma'}{\delta'-\gamma'}. \tag{11}$$

We must calculate the quotient of angles that appears here with consideration of equations (6) and (10). According to Fig. 90, $\delta = \vartheta + \delta'$; thus it follows from equation (6) that

$$\vartheta = \frac{\varepsilon'}{\varepsilon+\varepsilon'}(\vartheta+\delta'), \quad \vartheta = \frac{\varepsilon'}{\varepsilon}\,\delta'.$$

Equation (10), written for the deformed shape of the ellipsoid, becomes

$$\gamma' : \delta' = A' : C'$$

and shows, with sufficient accuracy, that

$$\frac{\gamma'}{\delta'-\gamma'} = \frac{\delta'}{\delta'-\gamma'} = \frac{1}{\mathsf{E}} = \frac{1}{\varepsilon+\varepsilon'}.$$

Thus
$$\frac{\vartheta+\gamma'}{\delta'-\gamma'} = \frac{\varepsilon'}{\varepsilon}\frac{\delta'}{\delta'-\gamma'} + \frac{\gamma'}{\delta'-\gamma'} = \frac{1}{\varepsilon+\varepsilon'}\Big(\frac{\varepsilon'}{\varepsilon}+1\Big) = \frac{1}{\varepsilon}.$$

Our previous result for the precessional period T' (equation (11)) may thus be written as

$$T' = \frac{\tau}{\varepsilon}, \tag{12}$$

in which form it coincides with equation (8) above. In words,

The precessional period of a top of deformable material and spheroidal form is calculated not from the ellipticity of its deformed shape (which would be $\mathsf{E} = \varepsilon + \varepsilon'$*), but rather from the ellipticity of its original form before the rotation, which it would again take after the cessation of the rotation. It is thus independent of the elastic compliance of the material, and is equal, in particular, to the precessional period of an absolutely rigid top whose ellipticity coincides with the original ellipticity of the deformable top.*

We will return to this theorem in the following chapter on geophysical applications, and will take it as the starting point for the representation of the pole oscillations and the explanation of the Chandler period. The transference of the preceding result to the phenomena of the Earth is difficult only in so far as 1) the Earth is not, in its mass distribution, a homogeneous ellipsoid of rotation, but rather is denser at its center than at its surface, and 2) for a deformation of the Earth, the gravitational action of the individual elements upon each other are of essential importance in addition to the elastic forces. The first-named circumstance implies that all numerical exercises that we must later make will be burdened with an uncertainty that corresponds to the uncertainty in the assumption about the mass distribution in the interior. Concerning the difficulty named in the second place, it is to be emphasized that in the case of the Earth we will have to define the ellipticity denoted by ε' in the preceding as the ellipticity that a sphere of the elasticity, mean density, and size of the Earth would take under the *mutual* action of the elastic and the gravitational forces, if it would be set into rotation with the velocity of the daily Earth rotation.

It may be pointed out that our assumption in this section of a nearly spherical top is advantageous not only for the application to the body of

the Earth. One easily sees, rather, that a spheroidal mass, or, more generally speaking, a mass with a spheroidal ellipsoid of inertia, is particularly to be recommended for the consideration of the deforming effect of centrifugal forces.

§9. The elasticity of the support.

In the usual experiments, the elasticity of the support may influence the character of the motion of the top to a higher degree than the elasticity of the top itself. One very frequently notes a resonance of the support (a tabletop) that is clearly perceptible to the touch as well as the ear. In order to generate and maintain the oscillations of a tabletop, however, energy is necessary. This must be drawn at the expense of the kinetic energy of the top, and will be partly transformed into heat by friction in the interior of the table, and partly dissipated externally, in that the vibrations of the tabletop are continuously transfered to more distant objects (to the floor through the legs of the table, etc.). The motion of the top will therefore be damped by the resonance of the tabletop. In the following presentation, we will naturally disregard all other energy-dissipating forces (friction, etc.). Concerning the form of the tabletop oscillation, we wish to assume that it consists of a transverse plate vibration in which the support point somehow remains fixed to the plate, and each point of the plate oscillates in the vertical direction. In itself, a horizontal tabletop oscillation in which the table legs bend is also entirely possible. We wish to assume, however, that the first form of oscillation is primarily excited by our top, which appears to be in conformity with the usual experimental conditions.

In order to make the problem mathematically accessible, we mentally replace the resonating table by a single mass particle that may move in the vertical direction about its natural equilibrium position O, and is drawn back to this mean position by a force that is proportional to the distance from O. The magnitude of this force and the magnitude of the mass are established by an experiment on the tabletop in the following manner: one measures the vertical bending deflection ζ of the tabletop at the position of the support point of the top on the basis of a load K, and calculates, in that one assumes proportionality between the load

and the deflection, the force k that corresponds to the bending deflection $\zeta = 1$ (say 1 cm). One further determines the period τ of a free oscillation of the tabletop, and thus calculates $m = \frac{\tau^2 k}{4\pi^2}$ as the "reduced oscillating mass" of the tabletop. This reduced mass is the mass of the material particle that we substitute for the support; the force that draws it back to its mean position O is $-k\zeta$. For the following, however, it is indispensable to include the damping of the tabletop oscillation, which is produced partly by energy conversion in the interior of the table and partly by external energy dissipation, since the consumption of the kinetic energy of the top that interests us depends directly on this damping. We therefore imagine that the logarithmic decrement of the tabletop oscillation is also determined, and call this $\frac{h\tau}{2m}$. We then ascribe to our mass particle a second force that is proportional and opposite to its velocity; it is equal, namely, to $-h\zeta'$. The free oscillation of our mass particle will then be similar in all respects to the free oscillation of the tabletop. They are determined by the simple equation

$$m\zeta'' + h\zeta' + k\zeta = 0. \tag{1}$$

The introduction of the quantities m, h, and k corresponds, in general, to speaking of the first term of this differential equation as the inertia, the second term as the damping, and the third term as the elasticity of the tabletop. If we wish to have a schematic image of the mass particle that we have substituted for the tabletop, then we can imagine, for example, the following apparatus: a massless spiral spring with a vertical axis is fixed at its lower end to an incompliant support, and carries at the upper end the mass particle m. The spring is restrained from lateral deflection by a guide bushing, but can be elongated or compressed in the vertical direction. The elongation or compression 1 is resisted with the force $\mp k$; in addition, a resistance acts in the interior of the spring or the guide bushing that is proportional to the velocity, and has magnitude $-h$ for velocity 1. The mass particle that is fixed to the upper end of the spring serves, in its turn, as the support of the lower end of the axis of the top.

The motion of the top, however, produces not the free oscillation described by the preceding differential equation for our mass particle (tabletop), but rather a certain forced oscillation. If R denotes the

reaction force of the moving top on the support in the vertical direction, then the equation for this forced oscillation is evidently

$$m\zeta'' + h\zeta' + k\zeta = R. \tag{1'}$$

The magnitude of R follows from the general impulse theorems, and, in particular, from the theorem for the vertical velocity of the center of mass of the top. If z denotes the vertical coordinate of the center of mass, measured from the fixed spatial point O, then the vertical component of the single-impulse (pushing-impulse) will be Mz'. Its rate of change is equal to the sum of the vertical forces acting on the top; that is, equal to the weight $-Mg$, where M is the mass of the top, and the reaction force $-R$ of the support against the top. One thus has the equation

$$\frac{d}{dz}Mz' = -Mg - R,$$

or

$$R = -M(g + z''), \tag{2}$$

as in equation (3) on page 515 of the appendix to Chap. VI. Equation (1′) thus becomes

$$m\zeta'' + h\zeta' + k\zeta + Mz'' + Mg = 0. \tag{3}$$

We distinguish further between the fixed spatial point O (the natural position of our mass particle m) and the moving point P (its instantaneous position at time t, which coincides with the instantaneous contact point of the top on the support), so that OP is equal to ζ. The figure axis, the line of nodes, etc., emanate from P; we will measure the Euler angles φ, ψ, ϑ with respect to this point. If E denotes the distance PS from the support point to the center of mass, then the vertical coordinate z of the center of mass is

$$z = \zeta + E\cos\vartheta. \tag{4}$$

Thus equation (3) can also be written as

$$(m + M)\zeta'' + h\zeta' + k\zeta + ME\frac{d^2}{dt^2}\cos\vartheta + Mg = 0. \tag{5}$$

One recognizes from this equation how the motion of the top is "coupled" to the motion of our mass particle m by means of the reaction force R.

In order to obtain the complete equations of motion of the problem, we must now consider the rotation of the top about the (vertically moving) support point P. Only gravity comes into consideration here as an external force, and gives the moment $MgE\sin\vartheta$ about the line of nodes; the reaction force R has zero moment with respect to the point P.

The change of the turning-impulse is thus determined. The calculation of the components of the turning-impulse is accomplished according to the rule of the Lagrange equations: one forms the expression for the *vis viva* and determines the impulse coordinates by differentiation with respect to the velocity components.

Since the point P is mobile, the expression for the *vis viva* is different from the usual expression. We place at the moving point P a coordinate system $x_1y_1z_1$ that is parallel to the coordinate system xyz at the fixed point O. The coordinates of any mass element Δm of the top then satisfy the equations

$$x = x_1, \quad y = y_1, \quad z = z_1 + \zeta,$$

so that

$$\frac{\Delta m}{2}(x'^2 + y'^2 + z'^2) = \frac{\Delta m}{2}(x_1'^2 + y_1'^2 + z_1'^2) + \frac{\Delta m}{2}\zeta'^2 + \Delta m z_1'\zeta'.$$

If one sums over the entire mass of the top, one may take ζ' in front of the summation sign. One thus obtains

$$T = T_1 + \frac{M}{2}\zeta'^2 + \zeta' \sum \Delta m z_1'.$$

Here T_1 is the *vis viva*

$$T_1 = \frac{A}{2}\,(\vartheta'^2 + \psi'^2 \sin^2\vartheta) + \frac{C}{2}\,(\varphi' + \psi'\cos\vartheta)^2$$

of the top with a fixed support point.

Furthermore, $\sum \Delta m z_1$ signifies the vertical coordinate of the center of mass in the $x_1y_1z_1$ system multiplied by the total mass of the top; one thus has, as in equation (4),

$$\sum \Delta m z_1 = ME\cos\vartheta, \quad \sum \Delta m z_1' = ME\frac{d}{dt}\cos\vartheta.$$

The expression for the *vis viva* therefore becomes

$$(6) \quad T = \frac{A}{2}(\vartheta'^2 + \sin^2\vartheta\,\psi'^2) + \frac{C}{2}(\varphi' + \cos\vartheta\,\psi')^2 + \frac{M}{2}\zeta'^2 - ME\zeta'\vartheta'\sin\vartheta.$$

If one now denotes by N, n, and $[\Theta]$ the three impulse components with respect to the coordinates φ, ψ, and ϑ, respectively, then one finds

$$N = \frac{\partial T}{\partial \varphi'} = C(\varphi' + \cos\vartheta\,\psi'), \quad n = \frac{\partial T}{\partial \psi'} = A\sin^2\vartheta\,\psi' + \cos\vartheta N,$$
$$[\Theta] = \frac{\partial T}{\partial \vartheta'} = A\vartheta' - ME\zeta'\sin\vartheta.$$

The first two impulse components have the same values as in the case of the fixed support point. If one writes the Lagrange equations for these components in the form

$$\frac{d}{dt}\frac{\partial T}{\partial \varphi'} - \frac{\partial T}{\partial \varphi} = 0, \quad \text{etc.},$$

then one finds, as previously,

$$N = \text{const.}, \quad n = \text{const.}$$

The third Lagrange equation, in contrast, formed according to the schema

$$\frac{d}{dt}\frac{\partial T}{\partial \vartheta'} - \frac{\partial T}{\partial \vartheta} = MgE\sin\vartheta,$$

is now

$$A\vartheta'' - ME\zeta''\sin\vartheta - ME\zeta'\vartheta'\cos\vartheta - \frac{\partial T_1}{\partial \vartheta} + ME\zeta'\vartheta'\cos\vartheta = MgE\sin\vartheta.$$

The value of $\partial T_1/\partial\vartheta$ has been calculated in a convenient form, for example, in equation (4) of §4 of this chapter. If one inserts this result into the preceding equation, cancels the two equal terms on the left-hand side, and divides by $\sin\vartheta$, there follows

$$\text{(8)} \qquad \frac{A\vartheta''}{\sin\vartheta} + \frac{(N - n\cos\vartheta)(n - N\cos\vartheta)}{A\sin^4\vartheta} = ME(g + \zeta'').$$

The effect of the mobility of the support point, or, as we can say, its "coupling" with the flexible support, is simply that the acceleration of the support point is added to the gravitational acceleration g on the right-hand side of our equation. The motion about the vertically moving support point is therefore just the same as the motion for a fixed support point, if one imagines, in the latter case, the variable force $M\zeta''$ applied in addition to the gravitational force Mg at the center of gravity. If we develop this thought somewhat further and reverse it to a certain extent, then we can say that the motion of the heavy top about a fixed support point is just the same as the motion of a body not subjected to gravity, whose support point is led in a straight line with constant acceleration g.

In any case, equations (8) and (5) contain the complete analytic formulation of our problem, and provide the required information concerning the mutual coupling of the two moving systems, the support and top. We note that our problem originally had four degrees of freedom, corresponding to the four position coordinates ζ, φ, ψ, ϑ. By means of the two impulse equations $n = \text{const.}$ and $N = \text{const.}$, two degrees of freedom are, to a certain extent, eliminated, so that we have left for the following only the two unknowns ϑ and ζ, and the two equations of motion (5) and (8).

Moreover, we could also have formed equation (5) according to the schema of the Lagrange equations if we had started from the complete *vis viva* $T^* = T + \frac{m}{2}\zeta'^2$ of our coupled system, and had correspondingly

formed the Lagrange equation

$$\frac{d}{dt}\frac{\partial T^*}{\partial \zeta'} - \frac{\partial T^*}{\partial \zeta} = K_\zeta.$$

The external force K_ζ that corresponds to the ζ-coordinate comprises the three terms $-k\zeta$, $-h\zeta'$, $-Mg$. One thus obtains

$$\frac{d}{dt}(M\zeta' - ME\vartheta' \sin\vartheta + m\zeta') = -h\zeta' - k\zeta - Mg,$$

which evidently coincides with (5).

It is now a matter of drawing further conclusions from equations (5) and (8). We let ourselves be guided here by the assumption of *small oscillations*. For what concerns the support, appearances show that this is always the case. For what concerns the top, our assumption signifies that we wish to restrict ourselves to motions with the character of pseudoregular precession. The variables ζ and ϑ then always differ little from certain mean values ζ_0 and ϑ_0, so that the differences

$$\mathsf{Z} = \zeta - \zeta_0, \quad \Theta = \vartheta - \vartheta_0$$

can be treated as small quantities. Whether the same holds for the differential quotients

$$\mathsf{Z}' = \zeta', \quad \Theta' = \vartheta', \quad \mathsf{Z}'', \quad \Theta''$$

we leave undecided, since for a rapid oscillation (and such is the case here) the differential quotients can be of a greater order of magnitude than Z and Θ themselves.

We first determine the appropriate mean values ζ_0 and ϑ_0. These are equal to the possible stationary values of our two coordinates, and are therefore equal to the values that are compatible with our equations under the assumption

$$\zeta' = \zeta'' = \vartheta' = \vartheta'' = 0.$$

According to equations (5) and (8), there follow

$$k\zeta_0 + Mg = 0, \tag{9}$$

$$(N - n\cos\vartheta_0)(n - N\cos\vartheta_0) = AP\sin^4\vartheta_0, \tag{10}$$

where ζ_0 signifies, as one sees, the constant deflection of the support under the influence of the weight of the top Mg and the elastic resistance k of the support. On the other hand, ϑ_0 is the inclination of the figure axis for which an exact regular precession is possible with given values of N, n, and $P = MgE$. In order to give ϑ_0 in more detail, we divide (10) by N^2 and assume that $\frac{AP}{N^2}$ is small for the pseudoregular

precession, and that the impulse falls nearly in the direction of the figure axis. One of the two factors

$$1 - \frac{n}{N}\cos\vartheta_0, \quad \frac{n}{N} - \cos\vartheta_0$$

on the left-hand side (namely the latter) must therefore be small; we set it equal to ε, find for the first factor the approximate value

$$1 - \frac{n}{N}\cos\vartheta_0 = \sin^2\vartheta_0 = 1 - \frac{n^2}{N^2},$$

and calculate from (10)

$$(11) \quad \varepsilon = \frac{AP}{N^2}\sin^2\vartheta_0 = \frac{AP}{N^2}\Big(1 - \frac{n^2}{N^2}\Big), \quad \cos\vartheta_0 = \frac{n}{N} - \frac{AP}{N^2}\Big(1 - \frac{n^2}{N^2}\Big).$$

This and only this inclination of the figure axis is compatible with an impulse that falls nearly in the direction of the figure axis, a complete immobility of the support, and the absence of an oscillation of the figure axis.

After the introduction of the new variables Z and Θ, equations (5) and (8) simplify, after the neglect of some obviously relatively small terms, to

$$(12) \qquad (m+M)\mathsf{Z}'' + h\mathsf{Z}' + k\mathsf{Z} = ME(\cos\vartheta_0\,\Theta'^2 + \sin\vartheta_0\,\Theta''),$$

$$(13) \qquad \frac{A\Theta''}{\sin\vartheta_0} + \Theta\frac{\partial}{\partial\vartheta_0}\frac{(n - N\cos\vartheta_0)(N - n\cos\vartheta_0)}{A\sin^4\vartheta_0} = ME\mathsf{Z}''.$$

The coefficient of Θ in (13) is still to be worked out. Since, according to (10), the expression to be differentiated is equal to P, one can write, under the application of the rule of logarithmic differentiation,

$$\frac{\partial}{\partial\vartheta_0}\frac{(n - N\cos\vartheta_0)(N - n\cos\vartheta_0)}{A\sin^4\vartheta_0} = P\Big\{\frac{N\sin\vartheta_0}{n - N\cos\vartheta_0} + \frac{n\sin\vartheta_0}{N - n\cos\vartheta_0} - \frac{4\cos\vartheta_0}{\sin\vartheta_0}\Big\}.$$

Here the first summand in the { } of the right-hand side is the essential term. According to (11), namely, this is equal to

$$\frac{\sin\vartheta_0}{\frac{n}{N} - \cos\vartheta_0} = \frac{\sin\vartheta_0}{\varepsilon} = \frac{N^2}{AP\sin\vartheta_0},$$

while the two remaining summands taken together give approximately $-3\cos\vartheta_0/\sin\vartheta_0$, which can, in contrast, be neglected. One can thus write equation (13) with sufficient accuracy as

$$(14) \qquad \Theta'' + \frac{N^2}{A^2}\Theta = \frac{ME}{A}\sin\vartheta_0\,\mathsf{Z}''.$$

This is a *linear differential equation with constant coefficients* in the two unknowns Θ and Z. Equation (12), in contrast, is *nonlinear*,

because of the term Θ'^2 on the right-hand side. The mathematical treatment of nonlinear equations, however, is very difficult; it is thus desirable to prove that we can strike the term Θ'^2 approximately.

We first consider our equations (12) and (14) under the assumption that the support is completely incompliant ($k = \infty$, $\mathsf{Z} = \mathsf{Z}' = \mathsf{Z}'' = 0$, $k\mathsf{Z}$ undetermined). Equation (14) then becomes

$$\Theta'' + \frac{N^2}{A^2}\,\Theta = 0,$$

and integrates to $\Theta = \begin{matrix} a\sin \\ b\cos \end{matrix}\left\{\frac{N}{A}t\right\}$, so that the oscillation period of the axis of the top is equal to $2\pi\frac{A}{N}$, and will therefore be small for large N. (Cf. Chap. V, §2, equation (15), where the same period was found.) Equation (12) is meaningless in this case, since $k\mathsf{Z}$ is undetermined; in fact, that equation is dispensable for the determination of the motion.

For a somewhat more compliant support, the period and the form of the oscillation of the axis of the top will be approximately the same as for the completely rigid support. We can, in any case, adopt the value of Θ for a rigid support in order to estimate the order of magnitude of Θ'^2 and Θ'' in equation (12). We then recognize that we may not claim that Θ' or Θ'' will be small if Θ is small (that is, if the oscillation amplitudes a and b are small numbers), since by differentiation the large factor N/A or N^2/A^2 is added. We may well claim, however, that Θ'^2 *is small compared with* Θ'', since the sine or the cosine components have the ratio, in the mean, $a^2 : a$ or $b^2 : b$. While the term Θ'^2 cannot be taken as absolutely small, it is still negligible compared with the term Θ''. We thus conclude that its influence on the course of the motion is small, and correspondingly consider ourselves as justified in striking this term in equation (12).

We may now adopt for the following the two differential equations

$$(15)\qquad \left\{\begin{aligned} (M+m)\mathsf{Z}'' + h\mathsf{Z}' + k\mathsf{Z} &= ME\sin\vartheta_0\,\Theta'', \\ \Theta'' + \frac{N^2}{A^2}\,\Theta &= \frac{ME\sin\vartheta_0}{A}\,\mathsf{Z}''. \end{aligned}\right.$$

They are discussed according to the known rules that are always applied for the method of small oscillations (cf. Chap. V, §8).

One sets

$$\mathsf{Z} = Ce^{\lambda t}, \quad \Theta = Be^{\lambda t}, \tag{16}$$

and determines the ratio of the two oscillation amplitudes C and B, as well as the oscillation frequency λ, by insertion of the preceding values into equations (15). In order to have equidimensional amplitudes in the calculation, it is necessary to go over, for example, from the amplitude of the angle Θ to the amplitude of the center of gravity oscillation, or, still more conveniently, to the vertical projection of this amplitude. If B is the amplitude of Θ, the amplitude of the center of gravity motion is $E \cdot B$, and the vertical projection of this amplitude is $E \sin\vartheta_0 B$. We set this equal to

$$D = E \sin\vartheta_0 B. \tag{17}$$

Equations (15) now become, after the insertion of the values (16) and (17),

$$\begin{cases} (\lambda^2 + h'\lambda + k')C = \mu\lambda^2 D, \\ \left(\lambda^2 + \dfrac{N^2}{A^2}\right)D = \nu\lambda^2 C, \end{cases} \tag{18}$$

where the abbreviations

$$h' = \frac{h}{M+m}, \quad k' = \frac{k}{M+m}, \quad \mu = \frac{M}{M+m}, \quad \nu = \frac{ME^2\sin^2\vartheta_0}{A} \tag{19}$$

have been used.

One notes here that ν, just like μ, is a pure number, and indeed a proper fraction. In fact, the moment of inertia A at the support point is equal, according to a well-known theorem on moments of inertia, to the corresponding moment of inertia at the center of gravity augmented by ME^2; thus $ME^2 < A$ and $\nu < 1$.

It can be concluded from equations (18) that

$$\frac{C}{D} = \frac{\mu\lambda^2}{\lambda^2 + h'\lambda + k'} = \frac{1}{\nu\lambda^2}\left(\lambda^2 + \frac{N^2}{A^2}\right). \tag{20}$$

The second equality of (20) provides the determination of λ; λ is calculated as *the root of the equation of the fourth degree*

$$\mu\nu\lambda^4 = \left(\lambda^2 + \frac{N^2}{A^2}\right)(\lambda^2 + h'\lambda + k'), \tag{21}$$

so that one has for disposal four possible values of λ that we name $\lambda_1, \ldots, \lambda_4$, comprising two complex-conjugate pairs.

For the discussion of the roots, we begin from the natural assumption that the support is rather incompliant (k no longer ∞, but k quite large, in particular large compared with N^2/A^2). Then it is necessary, according to equation (21), that

either $$\lambda^2 + \frac{N^2}{A^2} \cdots \text{ is very small}$$

or $$\lambda \cdots \text{ is very large.}$$

To consider the first possibility, we set $\lambda^2 + N^2/A^2 = \varepsilon$, calculate the value of ε under the neglect of the higher powers of ε, and thus determine two roots of our equation, whose approximate values we call λ_1 and λ_2. We find

$$\varepsilon = \lambda^2 + \frac{N^2}{A^2} = \frac{\mu\nu \frac{N^4}{A^4}}{k' - \frac{N^2}{A^2} \pm ih'\frac{N}{A}},$$

or, since k' is very large compared with N^2/A^2 and the right-hand side is therefore small,

$$\text{(22)} \qquad \left.\begin{matrix}\lambda_1\\ \lambda_2\end{matrix}\right\} = \pm i\frac{N}{A}\left(1 - \frac{1}{2}\frac{\mu\nu}{Q}\frac{N^2}{A^2}\left\{k' - \frac{N^2}{A^2} \mp ih'\frac{N}{A}\right\}\right),$$

where we have used the abbreviation

$$Q = \left(k' - \frac{N^2}{A^2}\right)^2 + h'^2\frac{N^2}{A^2}.$$

We find the other two roots of our equation by pursuit of the assumption that λ is very large. We set, for example, $1/\lambda = \varepsilon'$, and neglect ε'^3 and ε'^4. From (21), with consideration that k' should be large compared with N^2/A^2, the value of ε' follows from the quadratic equation

$$1 - \mu\nu + h'\varepsilon' + k'\varepsilon'^2 = 0;$$

its solution is

$$\varepsilon' = -\frac{h'}{2k'} \pm i\sqrt{\frac{1-\mu\nu}{k'} - \frac{h'^2}{4k'^2}}.$$

Thus the two root values of λ are

$$\text{(23)} \qquad \left.\begin{matrix}\lambda_3\\ \lambda_4\end{matrix}\right\} = \frac{-h' \mp i\sqrt{4(1-\mu\nu)k' - h'^2}}{2(1-\mu\nu)}.$$

The root pair (λ_1, λ_2) may be compared to the slightly different root pair $\pm iN/A$ that corresponds (cf. page 615) to the oscillation of the axis of the top for a completely incompliant support. The values (λ_1, λ_2) differ from this, namely, by the real part

$$-\frac{1}{2}\frac{\mu\nu}{Q}\frac{N^4}{A^4}h',$$

which is seen to be a consequence of the damping effect of the support. The imaginary part is also somewhat modified by the resonance of the support. On the other hand, we can compare the root pair (λ_3, λ_4) with the root values that occur for the oscillation of the support,

according to equation (1), in the absence of the top. One also recognizes here a reverse action of the top on the oscillation of the support, a reverse action that moreover may be regarded most simply, as one easily sees, as an apparent increase of the mass m of the original oscillating system.

We can now conclude the following with respect to the character of the motion: in every case, the oscillations of both the axis of the top and the support must be composed additively from terms of the form

$$e^{\lambda_1 t}, \quad e^{\lambda_2 t}, \quad e^{\lambda_3 t}, \quad e^{\lambda_4 t},$$

with coefficients C_i and D_i whose ratios are determined by equation (20). The conjugate exponential quantities will thus be combined pairwise into trigonometric functions; each pair will define an oscillation number and a damping factor. In particular, the oscillation of the axis of the top consists, if we disregard the effect of the support, of a pure periodic oscillation with the oscillation number $N/2\pi A$. *Through the coupling with the support, this oscillation number is somewhat changed, and, moreover, the oscillation is damped, so that it must gradually expire; it is overlaid, however, with the second damped oscillation, which, under the the assumption of a rather incompliant support* ($k' > N^2/A^2$), *has an essentially higher oscillation number.* On the other hand, the natural oscillation of the support is, as long as we disregard the stimulating effect of the top on the support, a damped oscillation with a very large oscillation number. *Through the coupling with the top, its oscillation number and its damping will likewise be somewhat modified, and this oscillation will be overlaid with vibrations of a smaller oscillation number, whose period lies in the vicinity of the natural period of the axis of the top.*

It is easy to see that the slower of the two oscillations, which lies near the eigenoscillation of the axis of the top, will be expressed more clearly in the motion of the top than in the motion of the support, and, conversely, that the faster oscillation, which we have compared with the eigenoscillation of the support, will be more strongly apparent in the oscillation of the support than in those of top. In fact, equation (20) shows that for our first root pair $\lambda = \lambda_1$ or $\lambda = \lambda_2$, for which $\lambda^2 + N^2/A^2$ is small, C is also small compared with D, and, in contrast, that for the second pair $\lambda = \lambda_3$ or $\lambda = \lambda_4$, for which $\lambda^2 + h'\lambda + k'$ is equal,

as one easily calculates, to $\mu\nu\lambda^2$, C is equal to D/ν, and therefore must be larger than D. *Each of our two systems, the top and the support, oscillates more strongly in the period that is more natural to it.*

It is particularly evident in observations that the oscillation of the top that is coupled to the support is a damped oscillation. It is seen that the small oscillation of the pseudoregular precession soon expires, and that the stationary motion of pure regular precession ($\mathsf{Z} = 0$, $\Theta = 0$, or, written in terms of our earlier coordinates, $\zeta = \zeta_0$, $\vartheta = \vartheta_0$) is approached as the final state. Moreover, we saw previously that other dissipative influences (friction at the support point) also act in a similar manner to diminish the nutation and simplify the process of the motion. In every case, however, the resonance of the support plays an essential role for this phenomenon in the now explained sense.

§10. Appendix. The influence of friction on the top on the horizontal plane.

In the appendix to the previous chapter, we considered the motion of a top that moves freely on a horizontal plane, without regard of friction. We had to point out at the conclusion of that appendix, however, that the actually observed motion has only a distant similarity with the motion found there. The basis for this naturally lies in the fact that friction, which we there neglected, has not a secondary corrective significance, but is actually decisive for the character of the trajectory, as smooth as the surface (mirror-glass pane) may be.

Since the trajectory of the horizontally moving top may be observed particularly well (see below), and since its regular and beautiful form may claim a special interest, we wish to complete our previous deliberation by the consideration of friction, to the extent that it becomes applicable to the general explanation of the actual phenomena. We will abstain, however, from quantitative calculations in the sense of §4 and §5 of this chapter, and discuss the influence of friction only qualitatively; we further abstain from a renewed discussion of air resistance, etc., since these effects withdraw in importance compared with sliding friction.

A remark that we made previously for the effects of friction in general also applies to the present problem: apparently insignificant secondary

circumstances can strongly influence the character of the motion. It is generally not indifferent, for example, whether the tip that slides on the support is more or less sharp, whether the support is more or less uneven, etc. The effect of such circumstances becomes particularly significant for a phenomenon that we have often had occasion to observe when using a steel point, and that we may designate as the process of "rooting": the point of the top is caught in a nearly invisible depression of the support, in which it subsequently becomes fixed; the top no longer moves on the horizontal plane, but rather is constrained, as if by an unseen seat, to turn about a fixed point, as in our original top problem. Whether and how this rooting occurs may not be determined in advance. Only this much is clear a priori and confirmed by observation: a sharpened point will bore into the support more easily than a rounded point, which may slide over possibly present depressions, and a coarse and soft support (paper, and particularly pasteboard) is more favorable for the effect than a hard and smooth surface (glass pane). A smoked glass pane, on whose surface the top itself produces unevenness by accumulation of the soot, is again more favorable than an unsmoked pane, and rooting more frequently occurs for a nearly upright position of the figure axis than for a strongly inclined axis, since the required lateral forces for the stopping of the support point are smaller in the former case than in the latter. Finally, this and other irregularities in the motion can occur more easily as the dimensions and the mass of the top are smaller, and as the originally imparted impulse is smaller or has diminished more in the course of the motion. In the following we will exclude this phenomenon of rooting, about which little may be said theoretically; we therefore assume a sufficiently rounded point on a sufficiently plane and regular support.

We now wish to depict the general course of the motion, as it is observed under this restriction. In contrast to the results of our previous frictionless consideration, it is at first apparent to the eye that the horizontal projection of the center of gravity does not, as was previously claimed, move in a straight line with constant velocity (corresponding to the horizontal velocity initially imparted to the center of gravity), and that, when the initial horizontal velocity of the center of mass is zero,

the center of mass does not remain on a fixed vertical line, but rather describes a circular trajectory that approximately follows the trajectory of the support point on the plane. It is further apparent to the eye that the trajectory of the support point, which we previously described as a scalloped circle, has not a constant mean radius, but rather that its radius is generally diminished, although under some circumstances, particularly toward the end of the motion, its radius is occasionally increased. *The trajectories of the support point and the center of gravity* are now to be described as generally narrowing *spiral lines.* As a rule, the individual windings of the spiral lines lie not inside each another, but rather more or less next to each other, which acts very favorably for the clarity of the following figures. The spiral lines thus appear to be pulled laterally apart from one another in a certain direction. One could wish to regard this phenomenon as a consequence of an initial velocity originally imparted to the center of mass; experiments show in an unambiguous manner, however, that it is merely a matter of the effect of small inclinations and irregularities of the support. In fact, we can produce an arbitrarily strong pulling apart of the spiral line by a deliberate slanting of the support; the direction in which the windings of the spiral progress does not coincide with the direction of greatest inclination of the support, but rather deviates from this direction in a determined sense due to the action of the top. With respect to the angular velocity with which the successive circles of the trajectory are traversed (the "precessional velocity"), observation shows in an unambiguous manner that this velocity increases somewhat in the course of the motion, and that we therefore have a somewhat accelerated precession. Finally, we wish to mention as a general result of observation that the nutation of the axis of the top that produces the scalloping of the trajectory of the support point, and thus contributes much to the curiously interesting impression of the curve, is always of very small magnitude in experiments, so that it only inessentially interrupts the uniform course of the trajectory. While in the previous chapter we placed special emphasis on the nutation of the axis of the top and approximated it by trigonometric functions (in a more rigorous calculation, it would be represented by elliptic or hyperelliptic functions), we will now generally neglect this nutation in the discussion of the observations.

As evidence for the preceding description of the observed processes, we give in the adjacent figures two examples of support point trajectories for two different tops, both of which were automatically registered

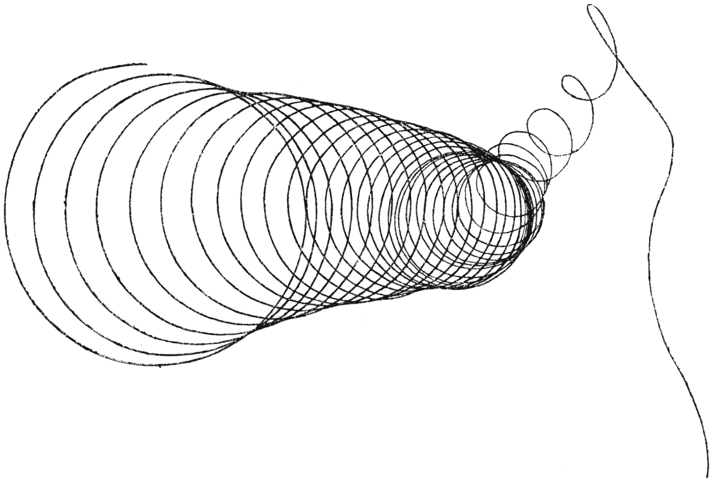

Fig. 91.

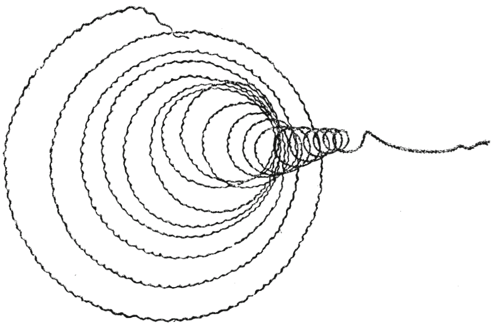

Fig. 92.

and then photographically reproduced, so that they are a direct documentation of the observation.

The first of these was most kindly placed at our disposal by L o r d K e l v i n. The lower end of his top, which he let run on drawing paper, was terminated by a lead pencil. We see here the gradual diminishment of the trajectory of the support point, as described above. The individual windings of the spiral trajectory run from left to right in the figure.

The diminishment of the radius of curvature of the trajectory continues in this example until the end, where the impulse is already strongly weakened and the guideline of the trajectory becomes somewhat uncertain and irregular. The curve finally runs in some irregular loops that correspond to the falling over of the top. An appropriate manner of construction for such a self-registering top has been reported by C. B a r u s.*) In our own study of these phenomena, we found it convenient to use a smoked mirror pane as a support, or, when we desired a stronger frictional effect, smoked writing paper, which is clearly marked by the point of the top. A rather light little clock wheel with an axle (the distance from the center of the wheel to the support point was about 1 cm, the diameter of the wheel was 5 cm, the weight of the wheel was 15 gr, and the steel support tip was more or less sharpened for different realizations) served us as a top. Our second figure was recorded on a smoked glass pane by such a top; the reproduction given here is the negative of the original, in which the trajectory is a light line lifted from the dark background of the smoke. Our second figure shows the nutation more clearly than the first, and, moreover, again allows the spiral form of the trajectory and a certain sidewise displacement to be recognized, which toward the end of the motion is apparent to the eye as a rather irregular run-out.

Having thus oriented ourselves objectively by experiments, we now proceed to the theoretical explanation of the observations.

Corresponding to the experimentally established insignificance of the nutation, we will make the simplifying assumption that the motion can be regarded at every instant as *precession-like*. A regular precession should now be understood as a motion in which the figure axis is inclined with respect to the vertical by a constant angle ϑ, and the center of gravity and the support point describe circles about the same vertical line with constant velocity. A motion will then be called precession-like if the inclination angle ϑ changes only slowly, and if the trajectories of the support point and the center of gravity are nearly circular and uniformly traversed spirals.

With respect to the form of the top at the support point, the repre-

*) Science, September 1896.[218]

sentation introduced for the top with a fixed support point (§3) may remain valid: the lower end of the top is a hemisphere of small radius ϱ; the lowest point of the hemisphere, which is not an individual point of the top, but rather changes at each instant, is the *support point* P. While the *midpoint* O of the hemisphere lying directly above P was a fixed point in §3, this point now describes a circle for regular precession. If we lay through P a plane perpendicular to the instantaneous rotation axis, then this plane cuts our hemisphere in a circle that we can call the "support circle"; the collected points of this circle will, in so far as the instantaneous axis of rotation does not change too rapidly in the top, successively assume the role of the support point, in that they are transformed by the rotation, one after another, into the position of the lowest point of the hemisphere.

As the law of friction—only sliding friction is treated—we will adopt the Coulomb law (§2). The frictional resistance W at the support point is then a horizontal force of magnitude μR, where R signifies the reaction force of the support on the top. This force is generally equal, as discussed on page 515, to $M(g+z'')$, where z'' signifies the acceleration of the center of gravity; for a precession-like motion we thus have, with sufficient precision,

$$R = Mg, \quad W = \mu Mg. \tag{1}$$

The direction and sense of the frictional resistance depend on the direction of the sliding at the support point. In order to determine the latter, we will temporarily choose the midpoint O of the named hemisphere as the "reference point," and decompose the motion of the top into a parallel displacement, whose velocity coincides with the velocity of the point O, and a rotation about an axis through O. The point P obtains, in this manner, the two velocities v and V; v is the velocity of the parallel displacement, or the velocity of O, and V is the velocity that P obtains due to the rotation about O. If, as we wish to assume, the instantaneous rotation axis nearly coincides with the figure axis, then the direction of V is nearly perpendicular to the figure axis, and the magnitude of V will be equal to the perpendicular distance from the point P to the figure axis (that is, $\varrho \sin\vartheta$) multiplied by the instantaneous rotational velocity of the top about O. The direction of the sliding will then be found by the geometric composition of the two velocities v and V;

by geometric and not algebraic composition, since, as we will see, the directions of v and V are necessarily inclined with respect to each other. One can now distinguish three cases; namely,

$$1)\ V > v, \quad 2)\ V = v, \quad 3)\ V < v.$$

Case 1) will be the usual case for a rapid rotation of the top; for a larger rotation, namely, the velocity V of the support point corresponding to the rotation will be larger, and, judging by the results of the frictionless motion, the precessional velocity and thus also the velocity v will be smaller. For a sufficiently strong rotation, one will indeed neglect v compared with V, and the direction of the sliding can be identified with the direction of V.

Case 3) will occur if, in the course of the motion, the eigenrotation is already considerably weakened by friction. The velocity v is then decisive for the determination of the sliding direction.

If we conceive the equation $V = v$ as a condition not only for the magnitudes but also for the directions (to be measured in the opposite sense) of the velocities v and V, then no sliding at all occurs in the boundary case 2). When v and V are equal in magnitude and opposite in direction, namely, the instantaneous support point is at rest relative to the support; the support circle then rolls without sliding on the support. Whether this boundary case is ever temporarily realized in the course of the motion is doubtful, however, and depends on the initial conditions.

We first investigate the usual case 1) in more detail.

In Fig. 93, we have drawn the circles that are approximately described, according to our assumption, by the center of gravity S and the point O in a precession-like motion. Both circles are drawn by perpendicular projection onto the horizontal plane that bears the top. The rotation occurs approximately about the figure axis in the *clockwise* sense. Let the projection of the turning-impulse at the center of gravity onto this axis be, correspondingly, $N > 0$. Then the precession of the support point also occurs in the *clockwise* sense as seen from above. We can take this precessional sense from our previous results under the neglect of friction. The angular velocity ψ' of the line of nodes, which likewise signifies the angular velocity with which the support point rotates about the (for our previous consideration fixed) vertical line through the center of gravity, has, in the mean, the value (see, for example, equation (31) of page 526)

(2) $$\psi' = \frac{P}{N} = \frac{MgE}{N},$$

and is therefore *positive* for positive N. The *sense* of the precessional motion can certainly not be reversed by friction. The direction of the arrow assigned to the trajectory of O in Fig. 93 is thus justified.[219]

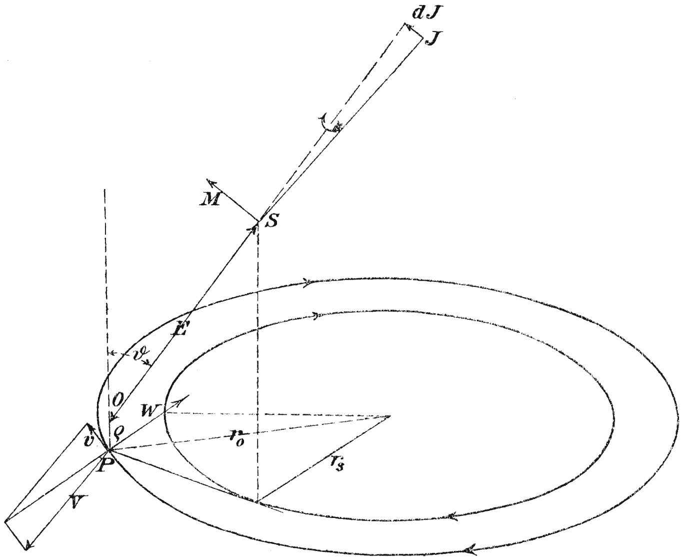

Fig. 93.

An arrow in the same direction evidently also applies to the trajectory of the center of gravity. We may, however, also carry over the *magnitude* of the precessional velocity given in (2) to the present case, since friction has only an indirect influence on this velocity (through the diminishment of N).

The previously introduced velocities v and V are now easily assignable in direction and magnitude. The velocity v has the direction of the tangent to the trajectory of O at the point P, and is directed in our figure from P toward the rear. The velocity V is approximately perpendicular to the figure axis, and is directed in our figure, for the established positive rotational sense, to the front. The magnitude of v is

(3) $$v = r_0\psi' = r_0\frac{MgE}{N},$$

where r_0 signifies the radius of the trajectory of O. The magnitude of the rotation about the figure axis follows from the eigenimpulse N, and is equal to N/C (C = the moment of inertia about the figure axis); thus approximately, according to the above,

(4) $$V = \varrho \sin\vartheta\, \frac{N}{C}.$$

Corresponding to the condition of case 1), we assume not only that V is greater than v, but also that V is large compared with v. This means, according to (3) and (4), that

$$\frac{N^2}{MgEC} \text{ is large compared with } \frac{r_0}{\varrho \sin\vartheta}. \tag{5}$$

For a sufficiently large eigenimpulse and a sufficiently inclined figure axis, this condition will in fact be satisfied.

Under the assumption (5), V and the frictional resistance W are simultaneously approximately perpendicular to the figure axis; W is directed in our figure toward the rear. The center of gravity motion, however, is most intimately related with the magnitude and direction of W. Since, namely, W is the single horizontal force that acts on the top, the horizontal acceleration of the center of gravity is parallel to W and is equal to $W/M = \mu g$ (see equation (1)). If, as we assume, the center of gravity approximately describes a circle with approximately constant velocity, then the center of gravity acceleration is approximately centripetal, and is therefore perpendicular to the circumference of the circle and toward the interior. Since this direction, on the other hand, is exactly parallel to the direction of W and thus approximately perpendicular to the direction of the figure axis, it follows *that the projection of the figure axis onto the horizontal plane that bears the top must be approximately tangent to the center of gravity circle.* And of the two tangents to the center of gravity circle that can be placed at the instantaneous position of P in Fig. 93, the tangent drawn to the front obviously corresponds to the properties in question for case 1). *The center of gravity always lags somewhat behind the support point in the traversal of its circle*; the figure axis does not intersect the vertical line through the midpoint of our circle, but rather *rotates obliquely* about the midpoint. The size of the center of gravity circle also follows easily from the magnitude of the center of gravity acceleration. The latter is, on the one hand, known to be equal to $r_s\psi'^2$; it was remarked above, on the other hand, that it is equal to μg. One thus has

$$r_s = \frac{\mu g}{\psi'^2} = \mu g\,\frac{N^2}{(MgE)^2}. \tag{6}$$

The center of gravity circle is always larger as the eigenimpulse is larger and the gravity moment $P = MgE$ is smaller; its size, moreover, naturally decreases with decreasing friction coefficient μ, and is reduced to zero for vanishing friction, in conformity with our previous results. The size of the concentric circle described by the support point is also now known. One has, according to Fig. 93,

$$r_0^2 = r_s^2 + (E \sin \vartheta)^2, \tag{7}$$

where $E \sin \vartheta$ signifies the projection of the length OS onto the horizontal plane. This circle will generally be only slightly larger than the center of gravity circle.

After the motion of the center of gravity is established, we must discuss the rotation about the center of gravity. In what sense will this be influenced by the frictional force W? We first construct the frictional moment $\mathfrak{M}(W)$ with respect to the center of gravity. If we neglect the distance ϱ between the points O and P, then the plane lying through S and W approximately contains the figure axis. The vector representing the frictional moment, which is to be constructed as the perpendicular to this plane, thus stands approximately perpendicular to the figure axis, and is, under the proportions of our Fig. 93, directed *upward* in the vertical plane through the figure axis. The vector of the frictional moment is now composed with the present turning impulse in such a manner that the impulse at each time element dt is changed by $dJ = \mathfrak{M}\, dt$. The impulse, which has the approximate direction of the figure axis, will thus be deflected upward. *The impulse is gradually uprighted by the action of friction.* In order to conclude that the figure axis is also uprighted, we recall the argument of page 555, according to which the rotation axis approximately follows the impulse axis, while the figure axis will be led rapidly around the rotation axis, so that its mean position approximately coincides with the position of the rotation axis. We thus recognize *that the figure axis always remains in the vicinity of the impulse, and is therefore likewise uprighted.*

The moment $\mathfrak{M}(R)$ of the reaction force is naturally to be considered in addition to the frictional moment $\mathfrak{M}(W)$; this moment has a horizontal axis, and indirectly produces, in the frictionless case, the precession of the top in a known manner.

In the first approximation, the magnitude of the impulse remains unchanged by friction, since the endpoint of the impulse progresses in a direction that is approximately perpendicular to the figure axis (cf. Fig. 93), and therefore is also approximately perpendicular to the direction of the impulse. It is clear, however, that the impulse must nevertheless be weakened with time. For, on the one hand, the uprighting of the figure axis performs work against gravity, and, on the other hand, frictional work is continually lost at the support. The loss of work must be defrayed by the *vis viva* of the top, and therefore in part by the *vis viva* of the center of gravity motion, and in part by the *vis viva* of the

rotational motion. The center of gravity velocity is equal to $r_s\psi'$, and has, according to equations (2) and (6), the magnitude

$$\mu g \frac{N}{MgE}.$$

Should this magnitude decrease, then N must decrease. We will obviously be led to the same result if we assume that the loss of work is made at the expense of the *vis viva* of the rotational motion. For the primary component of this *vis viva* is, as is well known, $N^2/2C$. *While the eigenimpulse N remains constant in the first approximation, it must, in the second approximation, slowly decrease.*

The gradual diminishment of N requires, according to equation (2), that the precessional velocity ψ' should accelerate, and further, according to equation (6), that the radius of the center of gravity circle should become smaller. It then follows from equation (7) that the radius r_0 of the circle described by the support point must also decrease; this radius will also be diminished to a smaller degree, moreover, by the uprighting of the figure axis (diminishment of the angle ϑ). This result coincides, as one sees, with the preceding results of observation.

We summarize our considerations as follows: *in case* 1) [$V > v$, or, better, V large compared with v] *the center of gravity traverses a circle whose radius gradually decreases, and therefore, more precisely said, traverses a diminishing spiral, and indeed with decreasing velocity. The same holds for the support point P or the hemisphere midpoint O. The figure axis, which is originally inclined to the vertical line at the midpoint of the center of gravity by the angle ϑ without passing through it, is continuously uprighted by the influence of friction in the course of the motion.*

We wish to discuss case 3) $V < v$ in a similar manner. Here the velocity v is decisive for the sense of sliding; under the conditions of our Fig. 94, where v is directed at the point P to the rear, the frictional resistance W will be directed to the front. If we make the assumption that the center of gravity moves nearly uniformly on a circle, then its centripetal acceleration must again be equal, in magnitude and direction, to W/M. The centripetal acceleration must therefore be directed, just like W, to the front in Fig. 94; that is, S must be found on the rear half-arc of the center of gravity circle. *The center of gravity now runs somewhat ahead of the support point in the direction of the motion.* The figure axis is once again inclined to the vertical line that passes through the midpoint of

the center of gravity circle, without passing through it. In contrast, we can no longer claim, as we did in case 1), that W is nearly perpendicular to the figure axis; the horizontal projection of the figure axis in Fig. 94 is not, as previously, a tangent, but rather a secant to the center of gravity circle. The size of the center of gravity circle is given, as previously, by equation (6).

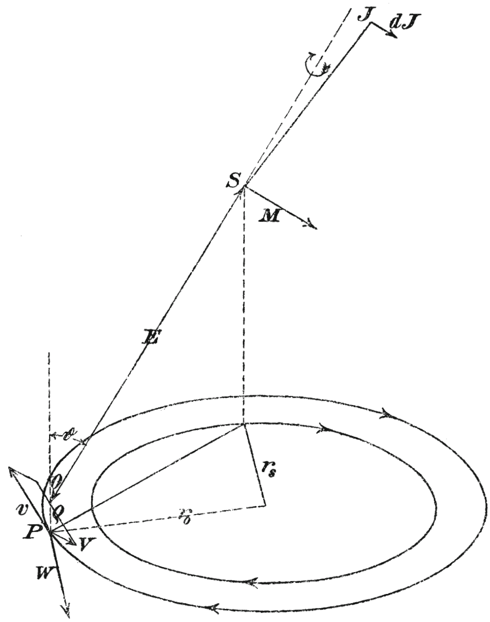

Fig. 94.

The influence of friction on the rotational motion is still to be considered. Here the conditions are opposite in sense to case 1). Since the frictional resistance in our figure is directed toward the front, it produces a moment at the center whose representing vector is slanted downward. Thus the end-point of the impulse will now be deflected downward by the action of friction. *The impulse vector and* (with the addition of the discussion of page 555) *the figure axis will descend.*

In order to consider the work relations in this case, we note that work is acquired by the descent of the figure axis, and that, in contrast, work is always expended by the frictional resistance. Presumably the latter quantity of work will dominate, so that, on the whole, the *vis viva*, and, in particular, the eigenimpulse will further decrease. We saw above that a diminishment of the center of gravity circle and a diminishment of the trajectory of the support point would follow. On the other hand, the increasing inclination of the figure would require an enlargement of the trajectory of the support point for an unchanged center of gravity trajectory. Which of the two circumstances will have more influence on the size of the trajectory of the support point remains undecided. In fact, sometimes an enlargement and sometimes a diminishment of the trajectory of the support point is observed when the eigenimpulse is weakened toward the end of the motion.

As the main result of this very uncertain consideration, however, it is to be emphasized that *the figure axis must descend in case* 3). Thus it cannot fail to occur that the upper parts of the top will ultimately touch the support, and that the top will come to rest after a few irregular motions.

We wish to consider the boundary case 2) $v = V$ very briefly. This case can occur only in passing and only under special circumstances, since we conceive the defining equation of the case as a condition for both the magnitudes and the directions of the velocities v and V. Since the support point does not slide on the support in this boundary case, the possibly present friction is to be designated as static friction (cf. §2). One then has, according to Coulomb, $W \leq \mu_0 Mg$, where μ_0 signifies the coefficient of static friction. It is possible, in particular, that the static friction will be zero if the support point, whose acceleration must also now be equal to W/M in direction and magnitude, is at rest, and therefore if the center of gravity circle has contracted to a point. In this case, it is conceivable that the top could execute its precession exactly as on an ideally smooth plane, which we assumed in the appendix to Chapter VI, and that the figure axis would neither rise nor fall. Such a motion could even persist arbitrarily long, if the other influences excluded from consideration (rolling friction, air resistance) would not destroy the condition 2) and cause the passage to case 3).

We have taken the distinction of the three cases $V > v$, $V = v$, and $V < v$ from a note by Archibald Smith,*) in which the influence of the particular form of the support end is also discussed. In order to classify our previous friction considerations into this distinction of cases, we note that $v = 0$ is naturally valid for the top with a fixed point O. Here we necessarily find ourselves under the condition of case 1). Correspondingly, we found previously that the figure axis of the top with a fixed point must always be uprighted due to sliding friction. A treatment of the present friction problem is also found, in so far as it concerns the rotation of the top about its center of gravity, in the well-known book of Jellett,**) but with the difference that the direction

*) Note on the theory of the spinning top. Cambridge Mathematical Journal, Vol. 1 (1846), p. 47.[220]

**) Theorie der Reibung, German by Lüroth and Schepp. Leipzig 1890, Chapter 8, p. 198.[221]

of the sliding is judged there from the velocity V alone, and the presence of the velocity v is ignored. Since Jellett thus sets, to a certain extent, the velocity v equal to zero, he likewise finds himself under the condition of case 1), and it is correspondingly shown, through calculations that can serve as a support to our qualitative deliberation above, that the figure axis must be uprighted. If one assumes, on the other hand, that the lower end of the top is an absolutely sharp conical cusp, then the support point (namely, this tip) will be a point of the figure axis; in this case $V = 0$ for a pure rotation about the figure axis, and we always find ourselves in case 3). Thus for an absolutely sharpened support point, the figure axis will always descend under the influence of friction.

The treatment given here is quite incomplete, both theoretically and experimentally. On the theoretical side, we have generally avoided writing the differential equations of motion with friction terms, since, due to the uncertainty of the physical basis, we could promise ourselves no benefit in the understanding of the actual observations that corresponds to the labor of thorough analytic developments. On the experimental side, we have been satisfied with the recording of the trajectory that is described by the contact point of the top on the support surface, but, in contrast, have been obliged to leave aside the more precise measurement of the impulse magnitude corresponding to this trajectory, the dependence of the motion on the initial conditions, the form of the support surface, etc. The latter failure appears to us to weigh more heavily than the former in the present case, since we wish, in general, to repeatedly emphasize that the understanding of the actual processes of the motion, in so far as frictional influences prevail, is to be advanced at least as much by observation as by calculation.

Chapter VIII.

Applications of the theory of the top.

Part A. Astronomical applications.

§1. The precession of the Earth's axis, treated in association with an idea of Gauſs.

Corresponding to the dominant position of astronomical applications in the older mathematical literature, the problem of the rotation of the Earth has been of conspicuous influence on the development of the theory of the top, as is proven, for example, by our borrowed nomenclature: regular precession, nutation, and line of nodes. We find the names of almost all the classical mathematicians associated with the history of this problem, beginning with Newton and continuing with Euler, d'Alembert, Laplace, Lagrange, and Poisson.

The theory of astronomical precession is very simple in the first approximation, and very complicated if an exhaustive treatment is attempted. The latter standpoint is taken in textbooks on astronomy;*) we must essentially adopt the former. To give the nonastronomical reader a glimpse of the laborious and admirable methods of astronomy, we present a few results of the more precise theory at the conclusion of this part of the chapter.

The difficulty increases enormously if we abandon the grounds of abstract dynamics and no longer regard the Earth as absolutely rigid. The debates that then occur are in no way closed at the present time. We will reserve this matter for the following part of the chapter, and first hold fast to the *assumption of rigidity.*

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*) We refer in the following to Tisserand, Mécanique céleste, t. II, Chaps. 22–27. In §194, p. 442, Tisserand reports on the history of the problem and the contributions of the named classical mathematicians to his research.[222]

F. Klein and A. Sommerfeld, *The Theory of the Top Volume III: Perturbations. Astronomical and Geophysical Applications*, DOI 10.1007/978-0-8176-4828-2_2,

Our method is modeled after a procedure given by G a u ſ s for the calculation of the secular perturbations of the planetary orbits. It has the advantage of great intuitiveness, and provides the individual components of the solution stepwise, according to the order of their importance. It appears not to have been applied to the present problem. Gauſs himself introduced his method with the remark that "the secular variations of a planetary orbit due to the perturbation of another planet are the same, whether the perturbing planet actually describes its elliptical orbit according to Kepler's law, or whether its mass is assumed to be distributed on the circumference of the ellipse in such a measure that equally large shares of the total mass are given to segments of the ellipse that are described in equally large times."*)

We wish to appropriate this idea and broaden it: we will distribute not only the mass of the perturbing body along its orbit, but also, where it is later desirable (§2), the mass of the perturbed body, which we will then treat as a rigid ring; we will learn to find not only the secular perturbations, but also, on the basis of a different mass distribution, the periodic perturbations (§3).

As Gauſs presented his method, it serves for the *exact* determination of the secular perturbations (at least those of the first order). In that we forgo the precision intended by Gauſs, we will simplify, in that we first disregard the eccentricities of the orbits; that is, for us, the orbits of the Sun and the Moon. We therefore assume that these orbits are circular. The nonuniformity of the mass distribution in the quotation of Gauſs, which indeed corresponds to the nonuniform motion on the ellipse, is then eliminated, and gives way to a uniform distribution on the circumference of the circle.

The most important element of the rotational phenomena of the Earth is its *precessional motion.* The approximate kinematic relations for this motion are already known (page 50): the axis of the Earth forms an angle of $23\,{}^1\!/_2{}^\circ$ with respect to the normal to the ecliptic (more precisely, at the present time, $23^\circ\,27'\,7''$, which number, however, is slowly changing), and rotates about the named normal at this angle once in approximately 26 000 years. Together with the daily rotation of the Earth, this motion of the axis represents a regular precession in the previous

*) Determinatio attractionis etc., Ges. W. Bd. 3, p. 331 and 357. It is this same treatise that contains the single direct communication of Gauſs on his theory of elliptic integrals.[223]

sense, and, in particular, a retrograde precession. If we consider the process from the side of the ecliptic toward which the north pole of the Earth points, then the rotation of the Earth about its figure axis occurs in the counterclockwise sense, and the rotation of the Earth's axis about the normal to the ecliptic occurs in the clockwise sense (see the adjacent figure; the three arrows that are assigned to the figure axis of the Earth F, the normal N, and the plane of the ecliptic E denote the direction of the Earth's rotation, the direction of the precession of the Earth's axis, and the apparent direction of the motion of the sun, respectively); the narrow polhode cone, whose size was determined on page 50, rolls in the interior of the herpolhode cone (cf. Fig. 8 of page 52, as well as Fig. 100a).

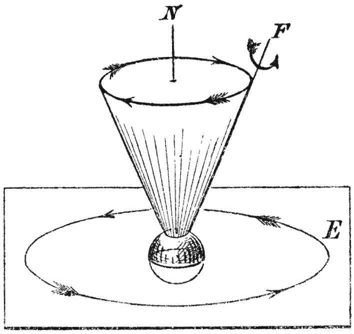


Fig. 95.

These relations, just like the number 26 000, are naturally not directly accessible to observation. The period of 26 000 years refers, rather, to the intersection points of the ecliptic with the equatorial plane, which are well known as the *vernal and autumnal equinox points* (*points of equal day and night*); the line that connects these points is the line of nodes K. It follows from the precessional motion of the Earth's axis that these points also rotate in the clockwise sense—that is, opposite to the sense of the apparent motion of the sun about the normal to the ecliptic—and indeed, as observation shows, by an amount of approximately $50''$ per year. The given approximate period of 26 000 years is calculated in reverse from this observation. The time of a complete revolution of the equinox points, and therefore the time in which the Earth's axis once encircles the normal to the ecliptic, is equal, namely, to

$$\frac{360^\circ}{50''} = \text{ca. } 26\,000 \text{ years.}$$

We now ask to what extent this phenomenon can be explained by our presently developed theory of the heavy symmetric top. That the precession is nothing other than an effect of gravity on the bulging toroidal mass of the rotating Earth was already recognized by N e w t o n,*) and supplied one of the most important and admirable verifications of his theory.[224]

*) Philosophiae naturalis principia mathematica. 1687. Book III, Prop. XXI, Theor. XVII.

Since the gravitational forces in question depend on the relative positions of the heavenly bodies, we may imagine the center of gravity of the Earth as fixed, and imagine the remaining heavenly bodies as moving with respect to the Earth. Of these bodies, we need only consider those that are distinguished by either their predominant size or small distance from the Earth; that is, only the Sun and the Moon. For a complete treatment of the rotational phenomena of the Earth, it would be necessary to consider the changing direction of the gravitational force due to the motion of the Sun and Moon in their orbits. We will return to the problem in this generality in the third section. We will there expand the temporally changing potential $V(t)$ of the Sun and Moon attractions into a trigonometric series with respect to the time t, and consider separately the terms that correspond to the orbital period of the Sun, the period of the lunar node motion, etc. The constant term of each series provides, in particular, the *secular effect* of the Sun and the Moon on the Earth, which gives as a resulting phenomenon the precessional motion of the Earth's axis that is of primary interest to us. We only indicate the more general consideration here; we now wish to use the intuitive procedure of Gauſs that directly separates the relevant secular component of the total attractive force.

We therefore imagine that the masses of the Sun and the Moon are distributed over their respective orbits with respect to the Earth, and, in particular, uniformly distributed, since we wish to assume that these orbits are circles. The radii of the circles correspond to the mean distances from the Earth to the Sun and the Moon. We must therefore investigate, instead of the actual Sun and Moon attractions, the attractions of the infinitely thin and uniformly dense "Sun-ring" and "Moon-ring." Further, we first wish to disregard the inclination of the Moon-ring to the ecliptic, which, as is well known, amounts to 5°, and imagine that the Moon-ring is rotated into the plane of the Sun-ring (see Fig. 96, where the customary astronomical signs ☉ for the Sun, ☾ for the Moon, and ♁ for the Earth are attached to the relevant rings). We also wish to make a simplifying assumption with regard to the nature of the Earth. We assume, as agreed, that it is rigid, and moreover is a body of revolution about the north–south axis with moments of inertia C and A, where, because of the bulge at the equator, $C > A$. For the calculation of all inertial effects, the particular form of the Earth in no way enters; any other body with the same moments of inertia C, A,

and A, placed in the position of the Earth, would behave exactly as the Earth does with respect to all inertial rotational effects. But more: we claim that the calculation of the attractive forces of the Sun and the

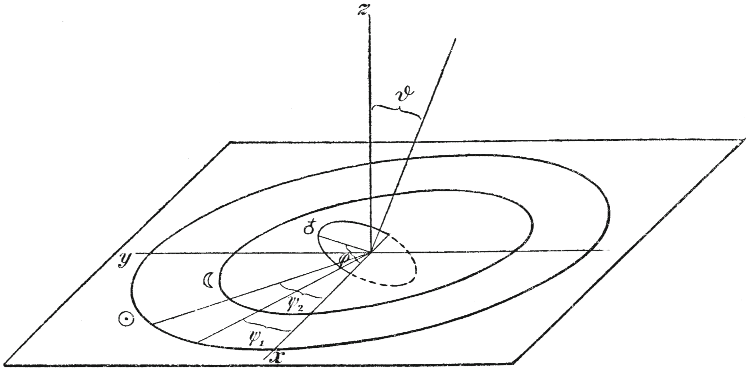


Fig. 96.

Moon also depends only on the values of the moments of inertia, in so far as we are satisfied with a certain approximation.

As a proof, we imagine the attractive potential of the actual Earth to an external, sufficiently distant point P; for example, a point on the Sun-ring or Moon-ring. This potential has the form $\sum \frac{m}{r}$, where m is a mass element of the Earth and the summation is extended over the entire mass of the Earth. Here one expands $1/r$ in powers of the ratios X/r_0, Y/r_0, and Z/r_0, where XYZ are understood as the coordinates of the mass element m, the origin of the coordinates is imagined at the midpoint (center of gravity) of the Earth, and r_0 signifies the distance of the point P from the center of the Earth. This series converges very rapidly, since the named ratios are at most equal, in our case, to the ratio of the radius of the Earth to the radius of the orbit of the Moon. One may thus truncate the series, if one strives for no great accuracy, with the terms of the lowest order. The terms of the first order vanish in the summation over the Earth, since the origin of the coordinates is chosen at the center of gravity. The terms of the second order give as coefficients, after the execution of the summations, the quantities $\sum mX^2$, $\sum mXY, \ldots$; that is, the moments of inertia and products of inertia (or centrifugal moments) of the Earth. If, in particular, one lets the coordinate axes coincide with the principal inertial axes, then the number of quadratic terms is reduced to three, and their coefficients will be the

three principal moments of inertia. Thus it follows, however, that in the first approximation (that is, for the consideration of merely the terms of the lowest order), all bodies with equal positions of the principal axes and equal values of the principal moments of inertia must also behave in the same manner with respect to gravitational effects. We can, in this respect, substitute for the Earth another arbitrary body, if only the ellipsoid of inertia of this body is identical to that of the Earth.

For many purposes, it is common and useful to imagine that the Earth is replaced by an ideal ellipsoid of revolution. In our case, however, another choice is preferred: we imagine a perfect, homogeneous sphere that is supplied at the equator with a belt of uniformly distributed mass. Let a be the moment of inertia of the sphere about one of its diameters and m be the mass distributed on our belt, the "Earth-ring." In order to have, for our purpose, a complete substitute for the actual Earth, we must arrange that the combination of the sphere and the ring possesses the same principal moments of inertia as the actual Earth. The moment of inertia of the ring about the north–south axis is mR^2, and its moment of inertia about an equatorial axis is $\frac{1}{2}\, mR^2$, understanding by R the radius of the Earth. Thus we must arrange that

$$mR^2 + a = C,$$

$$\frac{1}{2}\, mR^2 + a = A;$$

we must therefore choose

$$m = \frac{2(C-A)}{R^2}, \quad a = 2A - C. \tag{1}$$

It is clear from symmetry considerations that the sphere with moment of inertia a does not come into question in the calculation of the turning-moment of the attractive forces of the Sun-ring and the Moon-ring. Mechanical intuition immediately shows, moreover, that the Sun- and Moon-rings will strive to turn the Earth-ring in the plane of the ecliptic in the same manner. The relevant turning-force has the line of nodes as its axis, and acts about this axis in the clockwise sense as seen from the side of the line of nodes that bears the vernal equinox point, just as the gravity force does for a symmetric top whose center of gravity lies beneath the support point. We wish to calculate the magnitude of this turning-force.

Let m_1 be the mass and r_1 be the radius of the Sun-ring, and let ψ_1 be an angle, measured from the line of nodes, that distinguishes the individual points of the Sun-ring (see Fig. 96). The quantities m_2, r_2, ψ_2 have the analogous meanings for the Moon-ring. Finally, the same quantities for the equatorial Earth-ring are m (see above), R (Earth radius), and φ (an angle in the equatorial plane measured from the line of nodes).

The angle between the Earth-ring and the ecliptic is denoted by ϑ ($=$ ca. $23\,{}^1\!/_2{}^\circ$). We define rectangular coordinates x, y, z by letting the z-direction coincide with the normal to the ecliptic and the x-direction coincide with the line of nodes; we then have, for the Sun-ring and the Moon-ring,

$$x_1 = r_1 \cos\psi_1, \quad y_1 = r_1 \sin\psi_1, \quad z_1 = 0,$$
$$x_2 = r_2 \cos\psi_2, \quad y_2 = r_2 \sin\psi_2, \quad z_2 = 0,$$

while for the Earth-ring we have

$$x = R\cos\varphi, \quad y = R\sin\varphi\cos\vartheta, \quad z = R\sin\varphi\sin\vartheta.$$

In order to form the attractive potential of the Sun-ring on the Earth-ring, we calculate

$$\frac{1}{r} = \{(x_1-x)^2 + (y_1-y)^2 + (z_1-z)^2\}^{-\frac{1}{2}} = \{r_1^2 + R^2 - 2Rr_1 s\}^{-\frac{1}{2}},$$
$$s = \frac{xx_1 + yy_1 + zz_1}{Rr_1} = \cos\psi_1\cos\varphi + \sin\psi_1\sin\varphi\cos\vartheta,$$

and expand $\dfrac{1}{r}$ in powers of the small quantity $\dfrac{R}{r_1}$. If we write only the terms up to the second power inclusive, then

$$\frac{1}{r} = \frac{1}{r_1}\Big(1 + \frac{Rs}{r_1} + \frac{3}{2}\Big(\frac{Rs}{r_1}\Big)^2 - \frac{1}{2}\Big(\frac{R}{r_1}\Big)^2 + \cdots\Big).$$

This expression is to be integrated with respect to ψ_1 and φ; that is, over the Sun- and the Earth-rings. We find

$$\int_0^{2\pi} s\,d\psi_1 = 0, \qquad \int_0^{2\pi} s^2\,d\psi_1 = \pi(\cos^2\varphi + \sin^2\varphi\cos^2\vartheta),$$
$$\int_0^{2\pi} d\varphi \int_0^{2\pi} s^2\,d\psi_1 = \pi^2(1+\cos^2\vartheta).$$

Denoting the gravitational constant by f, the desired potential is therefore

$$V_1 = f\iint \frac{dm_1\,dm}{r} = f\,\frac{m_1 m}{(2\pi)^2}\int_0^{2\pi} d\psi_1 \int_0^{2\pi} \frac{1}{r}\,d\varphi$$
$$= f\,\frac{m_1 m}{r_1}\Big(1 + \frac{3}{8}\frac{R^2}{r_1^2}(1+\cos^2\vartheta) - \frac{1}{2}\frac{R^2}{r_1^2} + \cdots\Big).$$

This expression depends, as we see, only on the angle ϑ. The attractive force therefore acts only to change the angle ϑ; that is, acts only to turn about the line of nodes, as we already recognized above. The magnitude of this turning-force is, in the first approximation (that is, for the previously named omission of the higher powers of $\frac{R}{r_1}$),

$$\frac{\partial V_1}{\partial \vartheta} = -\frac{3}{4} f \frac{m_1 m R^2}{r_1^3} \sin\vartheta \cos\vartheta. \tag{2}$$

Finally, we express the mass m of the Earth-ring in terms of the moments of inertia A and C of the Earth (see equation (1)) and obtain

$$\frac{\partial V_1}{\partial \vartheta} = -\frac{3}{2} f \frac{m_1(C-A)}{r_1^3} \sin\vartheta \cos\vartheta. \tag{2'}$$

In the same manner, the turning-moment of the Moon-ring is

$$\frac{\partial V_2}{\partial \vartheta} = -\frac{3}{2} f \frac{m_2(C-A)}{r_2^3} \sin\vartheta \cos\vartheta. \tag{2''}$$

The desired turning-force is thus equal to the sum of these two expressions; that is, equal to

$$P \cos\vartheta \sin\vartheta,$$

where we have used the abbreviation

$$P = -\frac{3}{2} f(C-A)\left\{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right\}. \tag{3}$$

We have thus found a value for the external turning-force in the present case ($P\sin\vartheta\cos\vartheta$, $P < 0$) that is entirely analogous to the previous value for the heavy symmetric top whose center of gravity lies beneath the support point ($P\sin\vartheta$, $P < 0$).

We now make it clear to ourselves *that under the influence of this turning-force, regular precession represents, as previously, a possible form of motion.* At the same time, we note that regular precession is not, any more than it was previously, the most general possible form of motion. (The question whether the motion of the Earth is the particular *regular* precession or the general *pseudoregular* precession forms the proper subject of the following geophysical part of this chapter. In that we refer the reader to this second part, we will in the present part treat of the motion of the Earth, and likewise the motion of the Moon-ring, as a regular precession.) We rely most simply on the D'Alembert principle (Chap. III, §4): in every possible or "natural" motion of the top, the inertial action always maintains equilibrium with the external turning-force. The inertial action of the symmetric top for regular precession was found on page 175 to be

$$K = -C\mu\nu\sin\vartheta - (C-A)\nu^2\sin\vartheta\cos\vartheta; \tag{4}$$

this moment has the line of nodes as its axis, just as the external turning-force $P\sin\vartheta\cos\vartheta$ does in the present case. The stated principle therefore demands that

$$K + P\sin\vartheta\cos\vartheta = 0. \tag{5}$$

In equation (4), ν signifies the precessional velocity; that is, the angular velocity with which the axis of the Earth turns about the normal to the ecliptic; μ is the angular velocity of the Earth for its daily rotation, measured from the line of nodes. We must regard the quantity ν as unknown. Our equation yields two values for ν (as previously for the precessional motion of the symmetric top on page 178); since P (see below) is very small, one of these values will likewise be very small, and the other will be of the order of magnitude of μ. In our case, only the first value comes into consideration as the precessional velocity, since observations show unambiguously that ν is considerably smaller than μ. At the same time, the smallness of the ratio $\nu : \mu$ justifies us in neglecting the second term of equation (4) in comparison with the first, and in writing equation (5) more simply as

$$C\mu\nu = P\cos\vartheta. \tag{5$'$}$$

The theoretical value for ν is thus

$$\nu = \frac{P\cos\vartheta}{C\mu} = -\frac{3}{2}\frac{f}{\mu}\frac{C-A}{C}\left\{\frac{m_1}{r_1^3} + \frac{m_2}{r_2^3}\right\}\cos\vartheta. \tag{6}$$

The right-hand side may be formulated more conveniently for numerical calculation if we transform it with the help of Kepler's third law. The most precise expression of this law is the well-known equation

$$f\,\frac{(m+m')}{a^3} = \left(\frac{2\pi}{T}\right)^2;$$

here m and m' signify the two masses of the two-body problem, a the semi-major axis of the Kepler ellipse, and T the period. If we disregard the eccentricity, then a becomes identical with the mean distance r. There follows for the motion of the Earth around the Sun, since the mass of the Earth may be neglected in comparison with that of the Sun,

$$f\,\frac{m_1}{r_1^3} = \left(\frac{2\pi}{T_1}\right)^2, \tag{7}$$

and for the motion of the Moon about the Earth,

$$f\,\frac{M+m_2}{r_2^3} = \left(\frac{2\pi}{T_2}\right)^2, \quad \text{or} \quad f\,\frac{m_2}{r_2^3} = \frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2. \tag{7$'$}$$

Equation (6) is thus written as

$$\nu = -6\pi^2\,\frac{C-A}{\mu C}\left(\frac{1}{T_1^2} + \frac{m_2}{M+m_2}\frac{1}{T_2^2}\right)\cos\vartheta. \tag{6$'$}$$

We wish to draw some numerical conclusions from this formula. First, let us compare the component (ν_1) of the precession that is produced by the Sun with the component (ν_2) that is produced by the Moon. We have, evidently,

$$\frac{\nu_1}{\nu_2} = \Big(\frac{M}{m_2} + 1\Big)\frac{T_2^2}{T_1^2}.$$

Here $T_2 : T_1$ is the ratio of the (sidereal) period of the Moon to the (sidereal) year; that is, approximately $27\,{}^1\!/_3 : 365\,{}^1\!/_4$. For the ratio of the mass of the Earth to the mass of the Moon, we will adopt the value 82. As as result, there follows

$$\frac{\nu_1}{\nu_2} = 0{,}47, \quad \text{or} \quad \frac{\nu_2}{\nu_1} = 2{,}13.$$

The contribution of the Moon to the precession phenomenon is thus, because of its smaller distance and in spite of its smaller mass, more than twice as large as that of the Sun.

We now calculate the two components individually. We have

$$\nu_1 = -6\pi^2\,\frac{C-A}{C}\,\frac{\cos\vartheta}{\mu T_1^2}, \quad \nu_2 = 2{,}13\cdot\nu_1. \tag{8}$$

The quantity μ, the angular velocity of the rotation of the Earth, is equal to -2π divided by the length of the sidereal day,*) and μT_1 is therefore equal to -2π multiplied by the number of sidereal days in one year. This number is well known to be about 1 greater than the number of solar days in one year. Thus $\mu T_1 = -2\pi\cdot 366\,{}^1\!/_4$. (The negative sign depends on the fact that the rotation of the Earth occurs in the counterclockwise sense.) We must further know the value of $\dfrac{C-A}{C}$. Allowing ourselves to be guilty of a certain circular reasoning (see §4), we will accept for this ratio the value $\dfrac{1}{305}$. If we take the unit of time as the year, then the final result, expressed in arc seconds, is

$$\nu_1 = 3\pi\cdot\frac{\cos 23{,}5^\circ}{305\cdot 366\,{}^1\!/_4} = 16''. \tag{9}$$

*) This result is not entirely exact. Since the angular velocity μ, just like the Euler angle φ, whose temporal differential quotient it is, is measured with respect to the line of nodes and this line is displaced because of the precession in the opposite sense of the rotation of the Earth, μ will in reality be somewhat larger. The above result actually refers to the true angular velocity r, the third component of the rotation vector (p,q,r). Since, however, $r = \varphi' + \cos\vartheta\cdot\psi'$, and since further $\varphi' = \mu$ and $\psi' = \nu$, the difference between r and μ is equal to $\nu\cos\vartheta$, which quantity, because of the smallness of ν, does not come into consideration for our purpose.

The line of nodes thus rotates $16''$ *forward in the course of one year due to the Sun attraction alone.*

Further, it follows from equation (8) that

$$\nu_2 = 2{,}13 \cdot 16'' = 34''. \tag{9'}$$

Due to the Moon attraction alone, the line of nodes rotates $34''$ *in one year. The total amount of the precession is thus*

$$\nu_1 + \nu_2 = 50''. \tag{10}$$

So much for the explanation and approximate determination of the magnitude of the precession. For comparison with the sequel, we wish to write the motion of the Earth's axis in terms of the Euler angles ψ and ϑ. The present degree of approximation corresponds to the representation

$$\begin{cases} \psi = \psi_0 + 50'' \cdot t, \\ \vartheta = 23^\circ\, 27'\, 7''. \end{cases} \tag{11}$$

The quantity ψ_0 remains undetermined; it depends on the point of the ecliptic from which we measure the angle ψ.

§2. The regression of the lunar nodes. First extension of the Gaufsian method.

It is well known that the orbit of the Moon does not coincide exactly with the plane of the ecliptic; it forms, as previously mentioned, an angle of approximately 5° with respect to this plane (more precisely, an angle that oscillates between $5^\circ\, 0'$ and $5^\circ\, 18'$). The intersection points of the orbit with the ecliptic are the *lunar nodes*, and the line that connects these points is called the *line of nodes of the Moon*. This line of nodes now executes, under the influence of the Sun's attraction, a retrograde motion with respect to the sense of the motion of the Moon; the lunar line of nodes rotates about the normal to the ecliptic in the clockwise sense, just as the line of nodes of the Earth does, but with the considerably greater angular velocity of one rotation in approximately $18\,{}^2\!/_3$ years.

We can also bring this nodal motion into relation with our theory of the top, and can thus determine its numerical value. We must, however, assume essential points of the lunar theory as known. We must know from the outset, in particular, that the principal perturbation of the orbit of the Moon by the Sun consists of a motion of the lunar nodes without a change in the orbital inclination with respect to the ecliptic. We must further know that the (rather large) eccentricity of the orbit

of the Moon, which we will perforce disregard, will not appreciably influence the magnitude of the nodal motion, so that the nodal motion, on the one hand, and the perturbation of the Moon's orbit produced by its eccentricity, on the other hand, can be computed separately. Our consideration also wants, mathematically speaking, an existence proof for the motion of the lunar nodes; what we can obtain from the theory of the top is merely the calculation of the magnitude of this motion under the assumption of its existence.

We hold fast in the following to our previous representation of a Sun-ring and a Moon-ring, both of which we imagine as rigid and circular. The "Earth-ring," whose attraction we will consider after the fact, has too small a mass to come into perceptible consideration for our present purpose, so that we will first restrict ourselves to the attractive force of the Sun-ring. We imagine that the Moon-ring continuously rotates about its normal as a rigid body, with the velocity corresponding to the orbital motion of the Moon about the Earth. We are then faced with the following simple problem from the theory of the top: *the rotating Moon-ring stands under the influence of the attraction of the Sun-ring, which strives to draw it into the plane of the ecliptic. It describes, under the influence of this attraction, a regular precession about the normal to the ecliptic; what is its precessional velocity?*

With this formulation we have gone one step beyond Gauſs himself in the application of the Gauſsian method. While Gauſs distributed only the mass of the perturbing (the attracting) body on its orbit, we have also replaced the mass of the perturbed (the attracted) body by a continuous mass distribution. While, however, it is indifferent whether we think of the attracting mass (the Sun-ring) as in motion or at rest, it is essential that we consider the attracted mass (the Moon-ring) as a rotating ring. For it is directly this rotational motion that is, according to the fundamental principles of the theory of the top, necessary to maintain the inclination of the Moon's orbit with respect to the ecliptic in the presence of the turning-moment exerted by the Sun-ring.

We first form the attractive potential of the Sun-ring on the Moon-ring, and thus derive the turning-force that acts about the line of nodes of the Moon-ring. This turning-force is, according to equation (2) of the previous section,

$$\frac{\partial V}{\partial \vartheta_2} = -\frac{3}{4} f \frac{m_1 m_2 r_2^2}{r_1^3} \sin\vartheta_2 \cos\vartheta_2; \tag{1}$$

in fact, we need only replace the quantities m, ϑ, and R that refer to the Earth-ring in the named equation by the quantities m_2, $\vartheta_2 = 5°$, and r_2 that refer to the Moon-ring. If we write the right-hand side of (1) as $P_2 \sin\vartheta_2 \cos\vartheta_2$, then, with consideration of equation (7) of the previous section,

$$P_2 = -\frac{3}{4} f \frac{m_1 m_2 r_2^2}{r_1^3} = -\frac{3}{4} m_2 r_2^2 \left(\frac{2\pi}{T_1}\right)^2 = -\frac{3\pi^2}{T_1^2} C_2, \tag{2}$$

where $C_2 = m_2 r_2^2$ now signifies the moment of inertia of the Moon-ring about its figure axis.

A possible precessional motion of the Moon-ring with a long period will again be defined with sufficient precision by equation (5′) of the previous section, which we write, understanding by N the unknown precessional velocity and by M the rotational velocity of the Moon-ring, as

$$C_2 \mathsf{M} \mathsf{N} = P_2 \cos\vartheta_2; \tag{3}$$

there follows

$$\mathsf{N} = -\frac{3\pi^2}{T_1^2 \mathsf{M}} \cos\vartheta_2. \tag{4}$$

Now M signifies the angular velocity of the Moon-ring with respect to its line of nodes; it is equal to the angular velocity with which, as seen from the Earth, the Moon progresses in its orbit with respect to the Moon nodes. The corresponding period is called the draconian period; it is equal to 27,2 days.*) Thus

$$\mathsf{M} = -\frac{2\pi}{27{,}2} \quad\text{and}\quad \mathsf{M} T_1 = -2\pi \frac{365{,}25}{27{,}2}.$$

If we take the year as the unit of time, then, expressed in degree measure,

$$\mathsf{N} = \frac{3}{2} \frac{27{,}2}{365{,}25} \cos 5° \cdot 180° = 20{,}0°. \tag{5}$$

This is the number of degrees that the lunar nodes would regress in a year; the complete period of the lunar nodes would thus amount to

$$\frac{360}{\mathsf{N}} = 18 \text{ years.} \tag{6}$$

The value given above was $18\,{}^2\!/_3$ years, or, more precisely, 6793 days; this corresponds to the more precise value of $19\,{}^1\!/_3{}°$ for N. The difference

*) The relation of this angular velocity to the true or sidereal angular velocity of the Moon is the same as the relation between μ and r above. If we denote the sidereal angular velocity (that is, the quantity 2π divided by the sidereal month) by R, then $R = \mathsf{M} + \mathsf{N} \cos 5°$.

cannot surprise us, considering the crudeness of our representation of the Moon-ring and our disregard of the eccentricity of the Moon's orbit.

We wish to determine in a supplementary manner the influence of the *Earth attraction* on the motion of the lunar nodes, at least in a rough approximation. It is clear that the Earth can disturb the plane of the Moon's orbit only in so far as the Earth deviates from a spherical form, and that, for the decomposition of the Earth into an "Earth-sphere" and an "Earth-ring" that was discussed in the previous section, only the Earth-ring is to be considered. This Earth-ring with mass m now seeks, just as the Sun-ring does, to rotate the Moon-ring in a plane, and therefore here in the plane of the equator of the Earth. We conclude, as above, that under the influence of this turning-moment and due to the angular velocity of the Moon-ring, a regular precession about the normal of the named plane, and therefore here about the north–south axis of the Earth, is a possible form of motion of the Moon-ring, where we disregard the eigenmotion of the Earth's axis investigated in the previous section. We wish to determine the precessional velocity and the period of this precession. In that we find, as was predicted from the small mass of the Earth-ring, that this precessional velocity is very small, and that the precessional period will thus be very long compared with the corresponding velocity and period of the lunar node motion caused by the Sun, it is shown that the effect of the Earth changes the lunar node motion only in a small manner and in a secular form, and that we may neglect the Earth attraction in the previous calculation of the motion of the lunar nodes. The type of this (very insignificant) change consists not in a simple acceleration or deceleration of the nodal motion that is effected by the Sun, but rather in a change of the inclination of the Moon orbit with respect to the ecliptic, since, as mentioned, the precessional motion effected by the Earth occurs about a different axis from that effected by the Sun.

The turning-moment of the Earth-ring on the Moon-ring depends on the angle of inclination of the Moon-ring with respect to the equatorial plane of the Earth. This angle changes due to the nodal motion that is effected by the Sun, and oscillates by $\pm 5°$ in $18\,{}^2\!/_3$ years. It is simplest and most natural to replace this inclination angle by its mean value; that is, by the angle $\vartheta = 23{,}5°$ at which the equatorial plane of the Earth is inclined to the ecliptic. In that we do this, we thus disregard, as in the first section, the inclination of the Moon's orbit

with respect to the ecliptic, and rather imagine that the Moon-ring rotates in the ecliptic.

We can now take the turning-moment of the Earth's attraction on the Moon-ring directly from equation ($2''$) of the previous section. The formula there signifies the turning-moment that the Moon-ring, rotating in the ecliptic, exerts on the Earth-ring. Equally as large, however, is the turning-moment now in question. If we set this moment equal to $P_2' \sin\vartheta \cos\vartheta$, then, according to the named equation,

$$P_2' = -\frac{3}{2} f \frac{m_2(C-A)}{r_2^3}.$$

We compare the product $P_2' \cos\vartheta$ with the product $P_2 \cos\vartheta_2$, understanding by P_2 the value given in equation (2) of this section. According to equation (3) of this section, the angular velocity with which the lunar nodes would rotate about the north–south axis of the Earth as a result of the Earth-ring is to the angular velocity with which it rotates in the ecliptic as a result of the Sun attraction as $P_2' \cos\vartheta$ is to $P_2 \cos\vartheta_2$. If, as above, we name the two velocities N' and N, then

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{P_2' \cos\vartheta}{P_2 \cos\vartheta_2} = \frac{2(C-A)}{m_1 r_2^2} \frac{r_1^3}{r_2^3} \frac{\cos\vartheta}{\cos\vartheta_2}.$$

According to Kepler's third law (equations (7) and ($7'$) of §1), we may set

$$\frac{r_1^3}{r_2^3} = \frac{m_1}{M+m_2} \frac{T_1^2}{T_2^2},$$

and thus obtain

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{2(C-A)}{(M+m_2)r_2^2} \frac{T_1^2}{T_2^2} \frac{\cos\vartheta}{\cos\vartheta_2} = 2\,\frac{C-A}{C} \frac{C}{(M+m_2)r_2^2} \frac{T_1^2}{T_2^2} \frac{\cos\vartheta}{\cos\vartheta_2}.$$

Here an approximate value of C will be used in the numerator of the third factor on the right. We momentarily regard the Earth as a sphere of uniform density, so that we may assume, according to a well-known formula, $C = \frac{2}{5} M R^2$; there follows, finally,

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{4}{5} \frac{C-A}{C} \frac{M}{M+m_2} \frac{R^2}{r_2^2} \frac{T_1^2}{T_2^2} \frac{\cos\vartheta}{\cos\vartheta_2}.$$

All the factors of this expression are known numbers. The ratio R/r_2, for example, is equal to ca. $1/60$, while the ratio $M/M+m_2$ can be taken with sufficient accuracy equal to 1. With the use of the previously given other numerical values, there follows

$$\frac{\mathsf{N}'}{\mathsf{N}} = \frac{4}{5} \frac{1}{305} \left(\frac{1}{60}\right)^2 \left(\frac{365{,}25}{27{,}3}\right)^2 \frac{\cos 23{,}5^\circ}{\cos 5^\circ} = 1{,}2 \cdot 10^{-4}.$$

The velocity N' is therefore extraordinarily small compared with the velocity N. Conversely, the precessional period corresponding to N' is extraordinarily large compared with the period of the motion of the lunar nodes in the ecliptic, which amounts to $18\,{}^2\!/_3$ years. The precessional period corresponding to N' would be, namely,

$$\frac{18\,{}^2\!/_3 \cdot 10^4}{1{,}2} = 156\,000 \text{ years.}$$

The magnitude of this number shows immediately that our consideration has only the significance of an *estimate*, and not a *permissible calculation*. For, on the one hand, the elements of the Moon's orbit change during the named time duration in a significant and not to be predetermined measure, while they were taken as constant in our calculation. On the other hand, the position of the Earth-ring in space indeed changes completely because of the nodal motion of the Earth, while in our calculation we must assume that the position of the Earth-ring and its applied turning-moment are invariable. This assumption is permissible only for a time duration that is small compared with the precession period (26 000 years) of the Earth nodes, and, in contrast, is completely indefensible for the time duration found here, which is even greater than 26 000 years.

Nevertheless, the present calculation proves as much as we wished to show: namely, that the lunar node motion effected by the Earth-ring is to be neglected, and that the Sun's attraction is to be considered as the decisive factor here.

§3. The astronomical nutation of the Earth's axis. Generalization of the Gaußian method to periodic perturbations.

As we now turn to the *nutation of the Earth's axis* that was discovered by Bradley in 1747,[225] we emphasize in advance that this "astronomical" nutation has nothing in common, in *kinetic* respects, with the motion previously designated as the nutation of the axis of the top. The nutation of the general theory of the top (cf., in particular, Chap. V, §2) is due to the initial state of the motion not corresponding exactly to a regular precession, and the fact that the figure axis generally describes a cone in space even in the absence of all external forces. The astronomical nutation, in contrast, has its origin in periodically changing forces that act on the rotating Earth, which naturally cause a synchronous periodic motion of the Earth's axis. In association with a

well-known and important general distinction in mechanics, we can say concisely that *the previous nutation was a free oscillation, and the present nutation is a forced oscillation.*

The similarity of the two motions, which may justify the choice of the same designation, is only of a *kinematic* nature. In both cases, the nutation is an oscillation that is very short with respect to the period of the precession. The period of the free nutation in the general theory of the top is $2\pi A/N$, and that of the precession is $2\pi N/P$ (see, for example, page 305, equations (13) and (15)), so that the ratio of the two periods is the often named quantity AP/N^2, which we may assume, as a rule, to be a small number (for example, $< 1/100$). On the other hand, the astronomical nutation arises from the motion of the lunar nodes, and thus has a period of $18\,{}^2\!/_3$ years, while the period of the precession of the Earth's axis was calculated as 26 000 years, so that the ratio of these two periods is also very small, even $< 1/1000$.

In order to be able to associate the theory of the astronomical nutation with our considerations thus far, we must first broaden our adopted method of Gauſs once more. In its original form, this method serves for the calculation of *secular perturbations.* We will see, however, that it will also provide, with a slight modification, for *periodic perturbations.*

We first formulate the problem of the rotation of the Earth in the most general manner. We have on the one hand the Earth, and on the other hand the Sun and the Moon, which describe known relative orbits about the Earth, and correspondingly exert time-varying gravitational attractions. One finds the totality of the attractive forces most simply by differentiating the *attractive potential* with respect to the coordinates. The potential is calculated, as it always is for perturbation problems, from the relative positions of the relevant bodies *under the preliminary disregard of the perturbations to be found in the course of the calculation itself.* Since the perturbations are, as a rule, small in proportion to the principal motion, only a small error will thus arise. If, in contrast, one would adopt the disturbed motion for the calculation of the attractive potential, then one would, in addition to the so-called perturbations of the first order that we seek in the following, determine at the same time the "perturbations of the second order." Even if one wishes to know the latter, a stepwise approach and a temporary restriction to the perturbations of the first order is always recommended.

In our case, we understand by the undisturbed motion of the Earth its uniform rotation about the figure axis that is inclined to the ecliptic.

One will naturally decompose the potential V of the Sun and Moon attraction on the Earth into periodic and aperiodic components. The periodic components of the Sun potential V_1 will have the period of one year, and the periodic components of the Moon potential V_2 will have partly the period of one month and partly the period of the lunar nodes, etc. *Harmonic analysis* provides a methodical means of separating these components from one another. As is well known, one finds the coefficients of the trigonometric series in the form of certain integrals. The *aperiodic part of* V_1 is equal to $\frac{1}{T_1}\int V_1(t)\,dt$, extended over the time of a complete Sun orbit. This formula, however, may be interpreted as the potential of the relative orbit of the Sun that is bestowed, in an appropriate manner, with mass. Let dm be the mass element that we assign to the orbital element ds that is traversed with velocity $\frac{ds}{dt}$. Since the potential $V_1(t)$ corresponds to the entire mass of the Sun m_1, the potential of the named mass element will be $\frac{dm}{m_1}V_1(t)$, and the total potential of the Sun orbit provided with mass will be $\frac{1}{m_1}\int V_1(t)\,dm$. If this potential is to agree with the named coefficient of the trigonometric series, the mass distribution must be arranged so that the mass element

$$dm = \frac{m_1\,dt}{T_1} \tag{1}$$

is assigned to the element ds of the orbit. The total mass borne by the orbit is then precisely equal to the total mass m_1 of the Sun. We thus have exactly the original starting point of Gauſs. If, moreover, the trajectory is assumed to be circular, then the mass distribution is uniform. This was our standpoint for the previous treatment of the precession, which in fact arises from the constant or mean components of the Sun and Moon attractions.[226]

We now consider the *periodic components*. In that we again argue for the Sun, let T_1/n be the considered period, understanding by n a whole number. The coefficients of the two terms in the trigonometric series with this period are

$$\frac{2}{T_1}\int V_1(t)\,\cos 2\pi\,\frac{nt}{T_1}\,dt,\qquad \frac{2}{T_1}\int V_1(t)\,\sin 2\pi\,\frac{nt}{T_1}\,dt. \tag{2}$$

We again conceive them as attractions of the Sun's orbit bestowed with mass, where now, however, the mass $m_1 \dfrac{2}{T_1} \begin{matrix}\cos\\ \sin\end{matrix}\, 2\pi \dfrac{nt}{T_1}\, dt$ is assigned to the orbital element ds, understanding by t and dt the time and the time interval, respectively, in which the element ds is traversed by the Sun. The total mass assigned to the distribution is now zero, since we must employ, in addition to the positive masses, equally many "negative" masses. The density is not uniform even for a circular orbit, but rather is harmonically variable. The adjacent schematic figures illustrate these conditions in the cases $n = 0$ and $n = 2$.

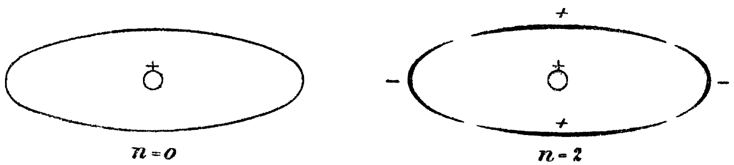


Fig. 97.

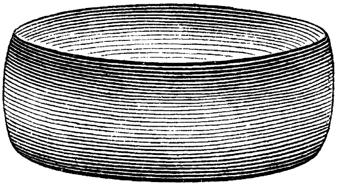
The periodic turning-forces acting on the body of the Earth result from the calculated trigonometric coefficients by differentiation with respect to the spatial coordinates and multiplication by $\begin{matrix}\sin\\ \cos\end{matrix} \dfrac{2\pi nt}{T_1}$. These turning-forces will produce perturbations of the Earth's axis with the same period T_1/n. We do not enter into this calculation here; it can be carried out according to the example to be given further below. The only perturbations of practical importance are the perturbations with the period $T_1/2$ and the perturbations caused by the Moon with the corresponding period $T_2/2$. The oscillation amplitude exceeds $1''$ for only one of these terms (see the formulas at the conclusion of the next section). The amplitudes of the terms with periods T_1, $T_1/3$, ... , T_2, $T_2/3$, ... are so small that they vanish even for the requirements of astronomical precision.

It is otherwise for the perturbations that have the period of the motion of the lunar nodes.

We first see how our method is to be formulated for these perturbations.

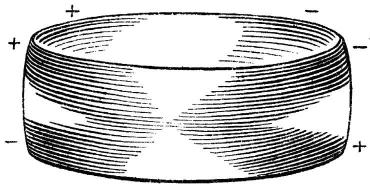
As we previously generated the Sun-ring and the Moon-ring by the simultaneous consideration of the changing positions of the Sun and Moon, we will now obtain a "Moon-ring surface" by representing the collected positions that the inclined Moon-ring takes in its precessional

motion. The Moon-ring surface that arises from the rotation of the Moon-ring about the normal to the ecliptic is evidently a doubly covered spherical zone of radius r_2 and height $2r_2 \sin 5°$.*) The following two figures represent the mass distributions that we assign to our Moon-ring surface in the cases $n = 0$ and $n = 1$. It may be explicitly emphasized that the intention of the introduction of our Moon-ring surface and the drawing of the following figures is nothing other than the primary intention of the Gaußian method: to illustrate the meaning of the calculations on a geometric basis. The calculations themselves are not fundamentally simplified, but rather are exactly the same as those that we would have to perform in a purely analytic procedure.



$n = 0$

Fig. 98 a.



$n = 1$

Fig. 98 b.

a) *In the case* $n = 0$ (secular perturbation), the mass distribution is to be chosen so that each element of the Moon-ring surface is assigned a mass $d\mu$, which, by analogy with equation (1), is equal to the product of the mass of the ring-element that sweeps out the element of the Moon-ring surface and the ratio dt/T, where dt is the sweeping time and T is the entire period of the lunar nodes. We wish to

*) We imagine the spherical zone as *doubly covered* (that is, consisting of *two shells* that are connected along their upper and lower edges) because each position of the spherical zone will be swept over *twice* by the rotating moon ring, once by the half-arc drawn in the foreground of Fig. 99, and once again by the half-arc that is not shown in the rear of this figure. The representation will be simplest if we ascribe a certain thickness to the spherical zone, conceive the exterior surface of the zone as one shell and the interior surface as the other, and stipulate that the Moon-ring passes over at each position from one to the other of the two shells at the upper or lower edge of the spherical zone. This is in conformity with the subsequent choice of our coordinates α, β; if, in the following, we integrate α and β from 0 to 2π, then we sweep over each position of the spherical zone twice, and therefore each of the two shells once; the one shell thus corresponds to coordinate values $-\pi/2 < \alpha < +\pi/2$, $0 < \beta < 2\pi$, and the other shell to the values $+\pi/2 < \alpha < +3\pi/2$, $0 < \beta < 2\pi$.

measure an angle α in the plane of the Moon-ring in such a manner that we calculate the lunar line of nodes OM (cf. Fig. 99) as $\alpha = 0$; each point P of the Moon-ring is then characterized by the central angle $\alpha = MOP$. On the other hand, we wish to designate as β the angle that the lunar line of nodes OM forms with an arbitrary fixed initial ray OA in the ecliptic; ψ is the angle that the nodal line of the Earth forms with the same ray OA. The angles α and β then represent skew-

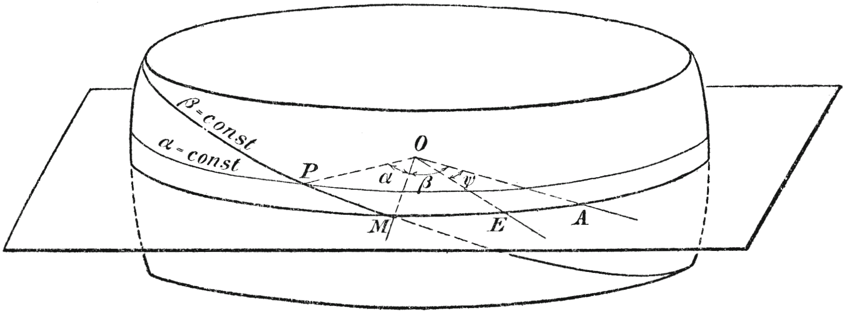


Fig. 99.

spherical coordinates on our spherical zone through which the position of any point of the spherical surface can be fixed, and which divide the spherical zone into parallelogram elements. The mass $d\mu$ allocated to such an element is now to be set equal to the product of the mass $m_2\, d\alpha/2\pi$ of the Moon-ring element that corresponds to the angle $d\alpha$ and the above-named ratio dt/T, which for a uniform rotation of the lunar nodes is equal to $d\beta/2\pi$; one thus has

$$d\mu = \frac{m_2}{4\pi^2}\, d\alpha\, d\beta. \tag{3}$$

The total mass assigned to the distribution, which follows from $d\mu$ by integration with respect to α and β, each between 0 and 2π, is naturally equal to the mass m_2 of the Moon-ring.

The density of the distribution—that is, the mass per unit area of the Moon-ring surface (calculated for both shells together)—is, as one easily understands from the inclined position of the Moon-ring, not uniformly disposed, but rather accumulates infinitely on the edges of the Moon-ring surface (for $\alpha = \pm\pi/2$). Along the latitude-circles, in contrast, the density is constant. It is attempted to indicate these relations in Fig. 98 by the strength of the hatching.[227]

b) *In the case* $n = 1$ (periodic perturbation), the assumed mass distribution on the Moon-ring surface is not uniform along the latitude-

circles, but rather is harmonically variable. The factor $2\cos\beta$ or $2\sin\beta$, namely, is added to the previously determined mass (cf. formula (2) for the coefficients of the trigonometric series). Thus

$$(4) \qquad d\mu = \frac{m_2}{2\pi^2} \begin{matrix}\cos\\ \sin\end{matrix} \beta \, d\alpha \, d\beta.$$

The total mass of the distribution, which is again obtained by integration with respect to α and β between 0 and 2π, is now equal to zero.

Here too the density, which we calculate as the algebraic sum of the mass per unit area of the two shells, accumulates toward the edges, and is oppositely equal in the neighboring octants of the spherical zone. These relations are indicated in Fig. 98b partly by the strength of the hatching, and partly by the addition of the signs.

After having thus explained Fig. 98, we form the potential that corresponds to the given assignments of mass; the potential that corresponds to the assignment (3) is called U, and the potentials that correspond to the assignments (4) are called w_1 and w_2 (w_1 corresponding to $\cos\beta$ and w_2 corresponding to $\sin\beta$). These potentials are nothing other than the first coefficients in the expansion of the attractive potential $V_2(t)$ exerted by the Moon on the Earth with respect to the period of the lunar nodes; the potential $V_2(t)$ is expressed, namely, in terms of U, w_1, w_2, and the angular velocity N of the lunar nodes as

$$V_2(t) = U + w_1 \cos \mathsf{N}t + w_2 \sin \mathsf{N}t + \cdots,$$

where we also write concisely

$$V_2(t) = U + W + \cdots, \quad W = w_1 \cos \mathsf{N}t + w_2 \sin \mathsf{N}t.$$

The constant term U corresponds to the value $n = 0$ of the index of the expansion, and the temporally variable term W comprises the two terms of the expansion that correspond to the value $n = 1$ of the index.

We cannot obtain anything essentially new from the value of U, but must rather return to the contribution of the Moon to the precessional motion of the Earth, which was calculated in the first section. We develop this calculation again only in order to convince ourselves that the previously neglected inclination of the Moon orbit with respect to the ecliptic does not essentially influence the phenomenon of the precession. The explanation and predictive calculation of the astronomical nutation will follow, in contrast, from the value of W.

The case $n = 0$. The potential of an element $d\mu$ of the Moon-ring surface on an element dm of the Earth-ring is, understanding by f the gravitational constant, $f\,d\mu\,dm/r$; thus the potential of the entire Moon-ring surface on the Earth-ring becomes

$$U = f \iint \frac{d\mu\,dm}{r}. \tag{5}$$

If x, y, z and x_2, y_2, z_2 are the coordinates of the points of the Earth-ring and the Moon-ring surface, respectively, then we set, as previously,

$$x = R\cos\varphi, \quad y = R\sin\varphi\cos\vartheta, \quad z = R\sin\varphi\sin\vartheta.$$

If, further, the nodal line of the Moon coincides directly with the nodal line of the Earth, then we can write, with respect to the same coordinate system,

$$x_2 = r_2\cos\alpha, \quad y_2 = r_2\sin\alpha\cos 5^\circ, \quad z_2 = r_2\sin\alpha\sin 5^\circ.$$

These coordinates correspond to the particular position $\beta = \psi$ of the Moon-ring (cf. Fig. 99). For arbitrary β, the value of z_2 remains as given, but the coordinates x_2, y_2 are obtained from the previous according to the coordinate transformation rule, where the rotation angle $\beta - \psi$ enters. The generally valid expressions are

$$\begin{aligned} x_2 &= r_2\big(\cos\alpha\cos(\beta-\psi) - \sin\alpha\cos 5^\circ\sin(\beta-\psi)\big),\\ y_2 &= r_2\big(\cos\alpha\sin(\beta-\psi) + \sin\alpha\cos 5^\circ\cos(\beta-\psi)\big),\\ z_2 &= r_2\sin\alpha\sin 5^\circ. \end{aligned}$$

We thus calculate

$$r^2 = (x-x_2)^2 + (y-y_2)^2 + (z-z_2)^2 = R^2 + r_2^2 - 2Rr_2 s,$$

where s signifies

$$\left\{ \begin{array}{l} \qquad\qquad s = \dfrac{xx_2 + yy_2 + zz_2}{Rr_2} = \\ \quad \cos\varphi\big(\cos\alpha\cos(\beta-\psi) - \sin\alpha\cos 5^\circ\sin(\beta-\psi)\big) \\ + \sin\varphi\cos\vartheta\big(\cos\alpha\sin(\beta-\psi) + \sin\alpha\cos 5^\circ\cos(\beta-\psi)\big) \\ + \sin\varphi\sin\vartheta\sin\alpha\sin 5^\circ. \end{array} \right. \tag{6}$$

By expansion in powers of r_2 there follows

$$\frac{1}{r} = \frac{1}{r_2}\Big(1 + \frac{R}{r_2}s - \frac{1}{2}\frac{R^2}{r_2^2} + \frac{3}{2}\frac{R^2}{r_2^2}s^2 + \cdots\Big). \tag{7}$$

We integrate this expression with respect to $d\mu$ and dm, in that we take $d\mu$ from (3) and set dm equal to $\frac{m}{2\pi}\,d\varphi$. First, $\int s\,d\varphi = 0$; further, the first and third terms on the right-hand side of (7) yield contributions to our potential that are free of the angles ϑ and ψ that determine

the position of the Earth-ring. Since we will later have to differentiate the potential with respect to these angles, these terms will also vanish. We thus do not write the first three terms or the higher terms of the expansion, and set

$$U = \cdots + \frac{3}{16} f \frac{mm_2}{\pi^3} \frac{R^2}{r_2^3} \cdot \int d\alpha \int d\beta \int d\varphi \, s^2 + \cdots .$$

One now easily calculates that

$$\int d\alpha \int d\beta \int d\varphi \, s^2 = \pi^3 \{(1 + \cos^2 \vartheta)(1 + \cos^2 5^\circ) + 2 \sin^2 \vartheta \sin^2 5^\circ\}.$$

Thus U becomes, if we introduce for the mass of the Earth-ring its value from equation (1) of §1,

$$U = \frac{3}{8} f \frac{m_2(C-A)}{r_2^3} \{(1 + \cos^2 \vartheta)(1 + \cos^2 5^\circ) + 2 \sin^2 \vartheta \sin^2 5^\circ\}.$$

The corresponding turning-moment on the Earth-ring is now found by differentiation with respect to ϑ, and is

$$\begin{aligned} \frac{\partial U}{\partial \vartheta} &= -\frac{3}{4} f \frac{m_2(C-A)}{r_2^3} \{1 + \cos^2 5^\circ - 2 \sin^2 5^\circ\} \sin \vartheta \cos \vartheta \\ &= -\frac{3}{2} f \frac{m_2(C-A)}{r_2^3} \left\{1 - \frac{3}{2} \sin^2 5^\circ\right\} \sin \vartheta \cos \vartheta. \end{aligned}$$

This value may be compared directly with the value derived in equation (2″) of the first section for the same turning-moment. It differs from that value, as one sees, only by the appearance of the additional factor

$$1 - \frac{3}{2} \sin^2 5^\circ = 1 - 0{,}012.$$

For numerical calculation, however, this difference plays no role, in so far as we wish to give, as in the first section, only the total seconds of the yearly precession. Thus the further treatment would follow exactly as there, and we can confirm all previous results as sufficiently precise when the inclination of the orbit of the Moon is considered.

b) *The case* $n = 1$. We also begin here from formula (5), where now, however, we understand by $d\mu$ the mass distribution defined by (4), and name the corresponding potentials, as agreed, w_1 and w_2. The quantity dm is, as above, equal to $\frac{m}{2\pi} d\varphi$, and the expansion (7) is inserted for $\frac{1}{r}$. In that we again suppress those terms that vanish in the integration or in the later differentiation with respect to ϑ and ψ, we write

$$\left.\begin{matrix} w_1 \\ w_2 \end{matrix}\right\} = \cdots + \frac{3}{8} f \frac{mm_2}{\pi^3} \frac{R^2}{r_2^3} \int \begin{matrix} \cos \\ \sin \end{matrix} \beta \, d\beta \int d\alpha \int d\varphi \, s^2 + \cdots .$$

If we first execute the integration with respect to α and φ, then we obtain from (6)

$$\int d\alpha \int d\varphi\, s^2 = \pi^2\{\cos^2(\beta-\psi) + \cos^2 5^\circ \sin^2(\beta-\psi) + \cos^2\vartheta \sin^2(\beta-\psi) + (\cos\vartheta\cos 5^\circ\cos(\beta-\psi) + \sin\vartheta\sin 5^\circ)^2\};$$

if we multiply this by $\cos\beta$ or $\sin\beta$ and integrate with respect to β, then all the terms vanish that are of odd dimension in $\frac{\cos}{\sin}\beta$ after the decomposition of $\frac{\cos}{\sin}(\beta-\psi)$. As the single nonvanishing term there remains

$$2\pi^2 \sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ \int \frac{\cos}{\sin}\beta\cos(\beta-\psi)\,d\beta = 2\pi^2\sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ \frac{\cos}{\sin}\psi.$$

Thus

$$\left.\begin{matrix} w_1 \\ w_2 \end{matrix}\right\} = \frac{3}{4} f\,\frac{m m_2 R^2}{r_2^3}\sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ \frac{\cos}{\sin}\psi.$$

The potential of the Moon-ring surface has thus been found for the two mass assignments that are schematically illustrated in Fig. 98b, or, as we can also say, for the two coefficients of the trigonometric expansion that correspond to the terms with the full period of the lunar nodes. The sum of these terms, which, as agreed, should be called W, now becomes

$$(8)\qquad W = \frac{3}{4} f\,\frac{m m_2 R^2}{r_2^3}\sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ\cos(\mathsf{N}t-\psi).$$

We transform this expression slightly, in that we consider, on the one hand, the definition of m (eqn. (1) of §1), and, on the other hand, the third Kepler law, and obtain

$$(9)\quad W = \frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\sin\vartheta\cos\vartheta\sin 5^\circ\cos 5^\circ\cos(\mathsf{N}t-\psi).$$

From the potential W, we now derive the turning-moment that acts on the Earth-ring. Since W depends on ϑ as well as ψ, we obtain by differentiation with respect to ϑ a turning-moment that acts about the line of nodes, and by differentiation with respect to ψ a second turning-moment that acts about the normal to the ecliptic. There follows, namely,

$$(10)\quad \left\{\begin{aligned} \frac{\partial W}{\partial\vartheta} &= \frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\cos 2\vartheta\,\sin 5^\circ\cos 5^\circ\cos(\mathsf{N}t-\psi), \\ \frac{\partial W}{\partial\psi} &= \frac{3}{4}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 (C-A)\sin 2\vartheta\,\sin 5^\circ\cos 5^\circ\sin(\mathsf{N}t-\psi). \end{aligned}\right.$$

We now see before us the following top problem: *the Earth stands under the influence of the just-named turning moments; what is its*

motion? For the further treatment of this problem, we naturally have to consider not the Earth-ring alone, as we did for the calculation of the attractive potential, but rather the entire body of the Earth.

The so-defined top problem differs from all previous problems in two respects: on the one hand, there is added to the turning-moment about the line of nodes, which is also present in the case of the usual heavy top, a second turning-moment about the "vertical" (here, the normal to the ecliptic). On the other hand, the two turning-moments vary not only with the position of the top, but also, rather, with time. The time dependence of this variability obviously determines the time dependence with which the Earth follows the turning moments. While the oscillation period for the previous *free* nutation that was investigated in the general theory of the top was determined by the mass distribution and the state of motion of the top itself, the period of the present *forced* nutation is prescribed by the alternation of the external force, and coincides, in our case, with the period of the lunar nodes.

In general, one can say that the problem of forced oscillations, if one disregards particular occurrences (resonance, etc.), is simpler than that of free oscillations, because the period of the oscillation need not first be discovered from the nature of the oscillating system, but rather is known from the outset. If the problem appears somewhat complicated in our case, this is due only to the combined character of the applied forces. Moreover, the method that we will adopt is exemplary for the treatment of any kind of forced oscillation, in case the oscillation is sufficiently small. The forced oscillation can always be superposed with the free oscillation, which we may, however, disregard in the present case, since we will speak of the possibility of such free oscillation in the second part of this chapter.

Mathematically speaking, the deferment of the free oscillation signifies that we will be satisfied with a *particular* integral of the present dynamic problem; namely, with the integral that is purely periodic with the period of the forcing, and for just this reason is called the *forced* oscillation. The *general* integral results by the addition of the most general *free* oscillation; that is, the general solution that corresponds to the force-free case. This addition is rigorous if the problem is governed by linear differential equations, and is valid with a certain degree of

approximation if, as in the present case, the differential equations of the problem can be reduced to linear equations by the neglect of small quantities.

For the calculation of the forced oscillation of the body of the Earth, we will use the *Lagrange equations in the coordinates* ϑ, ψ, φ. On the right-hand sides of these equations stand the components of the external force with respect to these coordinates; that is, in our case,

$$\frac{\partial W}{\partial \vartheta}, \quad \frac{\partial W}{\partial \psi}, \quad \frac{\partial W}{\partial \varphi} = 0.$$

The well-known expression for the *vis viva* is

$$T = \frac{A}{2}(\vartheta'^2 + \sin^2\vartheta\,\psi'^2) + \frac{C}{2}(\varphi' + \cos\vartheta\,\psi')^2,$$

which yields

$$\frac{\partial T}{\partial \vartheta} = A\sin\vartheta\cos\vartheta\,\psi'^2 - C(\varphi' + \cos\vartheta\,\psi')\sin\vartheta\,\psi', \quad \frac{\partial T}{\partial \psi} = \frac{\partial T}{\partial \varphi} = 0,$$

$$\frac{\partial T}{\partial \vartheta'} = [\Theta] = A\vartheta', \quad \frac{\partial T}{\partial \psi'} = [\Psi] = A\sin^2\vartheta\,\psi' + C\cos\vartheta(\varphi' + \cos\vartheta\,\psi'),$$

$$\frac{\partial T}{\partial \varphi'} = [\Phi] = C(\varphi' + \cos\vartheta\,\psi').$$

The Lagrange equations are now

$$A\vartheta'' - A\sin\vartheta\cos\vartheta\,\psi'^2 + C(\varphi' + \cos\vartheta\,\psi')\sin\vartheta\,\psi' = \frac{\partial W}{\partial \vartheta},$$

$$\frac{d}{dt}\big(A\sin^2\vartheta\,\psi' + C\cos\vartheta(\varphi' + \cos\vartheta\,\psi')\big) = \frac{\partial W}{\partial \psi},$$

while the third equation yields $[\Phi] = \text{const.}$ Since $[\Phi] = Cr$, where r is the rotational velocity of the Earth about its figure axis and $2\pi/r$ is the duration of the sidereal day, r also becomes constant, and thus the length of the sidereal day is not influenced by the presently considered lunar perturbations.

We introduce the angular velocity $r = \varphi' + \cos\vartheta\,\psi'$ into the previous equations, and write these equations more simply as

$$A\vartheta'' - A\sin\vartheta\cos\vartheta\,\psi'^2 + C\sin\vartheta\,r\,\psi' = \frac{\partial W}{\partial \vartheta},$$

$$\frac{d}{dt}(A\sin^2\vartheta\,\psi' + C\cos\vartheta\,r) = \frac{\partial W}{\partial \psi}.$$

We now consider that the angular rates ψ' and ϑ' are, according to observation, extraordinarily slower and have an extraordinarily smaller amplitude than the rotation r, and that r is therefore extraordinarily large compared with φ' and ϑ'. Correspondingly, we will strike all terms on the left-hand sides that do not possess r as a factor, and simplify the preceding equations as

$$C\sin\vartheta\, r\,\psi' = \frac{\partial W}{\partial\vartheta},$$
$$-\,C\sin\vartheta\, r\,\vartheta' = \frac{\partial W}{\partial\psi}.$$

If we insert on the right-hand sides the values from (10), there follow

$$\psi' = \quad \frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 \frac{C-A}{Cr}\sin 5^\circ\cos 5^\circ\,\frac{\cos 2\vartheta}{\sin\vartheta}\,\cos(\mathsf{N}t-\psi),$$
$$\vartheta' = -\,\frac{3}{2}\,\frac{m_2}{M+m_2}\left(\frac{2\pi}{T_2}\right)^2 \frac{C-A}{Cr}\sin 5^\circ\cos 5^\circ\ \cos\vartheta\ \sin(\mathsf{N}t-\psi).$$

Here we can once again make a simplification, in that we insert on the right-hand side the values of ψ and ϑ found in the first approximation (see equations (11) of §1); namely, $\psi = \psi_0 + 50''t = \psi_0 + \nu t$, $\vartheta = 23^\circ\,27'\,7'' = \vartheta_0$. The integration with respect to t may then be executed easily, and yields

(11)
$$\begin{cases} \vartheta = \dfrac{3}{2}\dfrac{m_2}{M+m_2}\left(\dfrac{2\pi}{T_2}\right)^2 \dfrac{C-A}{Cr}\dfrac{\sin 5^\circ\cos 5^\circ}{\mathsf{N}-\nu}\ \cos\vartheta_0\ \cos(\mathsf{N}t-\nu t-\psi_0), \\ \psi = \dfrac{3}{2}\dfrac{m_2}{M+m_2}\left(\dfrac{2\pi}{T_2}\right)^2 \dfrac{C-A}{Cr}\dfrac{\sin 5^\circ\cos 5^\circ}{\mathsf{N}-\nu}\ \dfrac{\cos 2\vartheta_0}{\sin\vartheta_0}\ \sin(\mathsf{N}t-\nu t-\psi_0). \end{cases}$$

In these equations, the theoretical representation of the astronomical nutation is achieved. As we see, both the angle ϑ and the angle ψ are subjected to a harmonic oscillation, whose period $2\pi/\mathsf{N}$ coincides with that of the lunar nodes. (We can, in particular, neglect the angular velocity ν of the Earth-nodes with respect to that of the lunar nodes N without further consideration.) In order to find the numerical value of the amplitudes, which may be called a and b, we first calculate

$$\frac{b}{a} = 2\,\mathrm{ctg}\,2\vartheta_0 = 1{,}9.$$

If we further take the year as the unit of time and use the previously given values

$$\frac{M}{m_2} = 82,\quad \frac{C-A}{C} = \frac{1}{305},\quad T_2 = \frac{27\,{}^1\!/_3}{365\,{}^1\!/_4},\quad r = -2\pi\cdot 366\,{}^1\!/_4,\quad \mathsf{N} = \frac{2\pi}{18\,{}^2\!/_3},$$

then the amplitude of ϑ, expressed in seconds, becomes

$$a = \frac{3}{2}\cdot\frac{1}{83}\cdot\frac{(365\,{}^1\!/_4)^2}{366\,{}^1\!/_4}\cdot\frac{18\,{}^2\!/_3}{(27\,{}^1\!/_3)^2}\cdot\frac{0{,}087\cdot 0{,}917}{305}\cdot\frac{360\cdot 60\cdot 60}{2\pi} = 9''.$$

There follows

$$b = 1{,}9\cdot a = 17''.$$

The axis of the Earth thus describes on the firmament a small ellipse that is called, after its discoverer, the B r a d l e y ellipse. The major axis of this ellipse amounts to $a = 9''$; it is directed toward the pole of

the ecliptic. An elementary geometric deliberation shows that the minor axis is $b \sin \vartheta_0 = 7''$.

We wish, finally, to supplement the representation of the motion of the axis of the Earth that was given at the conclusion of the first section (equation (11) of page 643) by the addition of the nutational terms. The representation is then

$$\left\{\begin{aligned} \psi &= \psi_0 + 50''\, t + 17'' \sin(\mathsf{N}t - \psi_0), \\ \vartheta &= 23^\circ\, 27' + 9'' \cos(\mathsf{N}t - \psi_0). \end{aligned}\right. \tag{12}$$

§4. Concluding remarks on the problem of precession and nutation. The determination of the mass of the Moon and the ellipticity of the Earth.

With the corrections considered until now, the subject is still far from closed. First, one can pursue further the influence of the lunar node motion, and calculate the terms of period $\frac{4\pi}{\mathsf{N}}, \frac{6\pi}{\mathsf{N}}$, etc. The first two of these terms are actually considered in practice, even though their amplitudes amount to only the tenth and fifth part of a second, respectively. Then, however, the eccentricities of the orbits of the Sun and, particularly, the Moon are to be considered. These eccentricities influence not only the periodic precession terms, but also the secular precession term. The resulting correction of the precessional velocity amounts, however, to less than $1''$.

We further wish to point out once again the previously discussed but not calculated influences that are caused by the changing position of the Sun and Moon in their orbits, and which have as their periods an aliquot part of the Sun's or the Moon's orbital period.

Finally, it is to be considered that all the elements that enter into our calculations, such as the eccentricity of the orbit of the Sun, the position of the ecliptic with respect to the fixed stars, etc., are subject to secular changes, changes that one develops in the usual manner as a series that progresses in powers of t. It follows, in particular, that the precessional angle is not simply proportional to time, but rather is represented, in its turn, by a power series in t. The coefficient of t^2 in this series is already extremely small, ca. $10^{-4} \cdot 1''$; nevertheless, its presence is sufficient to make results that refer to a large number of years and are inferred only from the first term (νt), as, for example, the calculation of the period of 26 000 years given at the beginning of this part of the chapter, appear illusory to a certain extent.

With consideration of these various influences, the formulas for the motion of the axis of the Earth become essentially more complicated. The precession is no longer uniform, but is rather somewhat accelerated or decelerated due to the latterly named conditions. In addition, a series of secondary nutations (for example, nutations with the half-period of the lunar nodes, the half-period of the Sun and Moon orbits, etc.) will be superposed on the primary nutation discussed so far. In order to give an illustration of the resulting formulas, we place as a counterpart to the approximate formulas at the conclusion of the previous section the following more complete description of the motion of the axis of the Earth. This is taken, with changes of notation, from the work of Tisserand;*) the origin and the meaning of the individual terms will be clear from the preceding.

$$\begin{aligned}
\psi = {} & 50'',37140\, t - 0'',00010881\, t^2 \\
& - 17'',251 \sin \mathsf{N}t + 0'',207 \sin 2\mathsf{N}t \\
& - 1'',269 \sin \frac{4\pi t}{T_1} - 0'',204 \sin \frac{4\pi t}{T_2}, \\
\vartheta = {} & 23^\circ\, 27'\, 32'',0 + 0'',00000719\, t^2 \\
& + 9'',223 \cos \mathsf{N}t - 0'',090 \cos 2\mathsf{N}t \\
& + 0'',551 \cos \frac{4\pi t}{T_1} + 0'',089 \cos \frac{4\pi t}{T_2}.
\end{aligned}$$

Even this more complete formula is not to be claimed as exact, and may be extended to an arbitrarily long time no more than our previous representation. Its purpose, rather, is only to make possible, under the current values of the astronomical constants, the predetermination of the position of the axis of the Earth for a time that is sufficiently long for the requirements of calculating astronomers. Other authors**) give still longer formulas.

At the conclusion of this part of the chapter, we must still discuss a certain circular conclusion of which we were guilty in the preceding numerical calculations. It concerns the *ratio M/m_2 of the mass of the Earth to the mass of the Moon*, and the so-called *ellipticity of the Earth* (concerning the terminology, cf. §7 of the present chapter); that is, the ratio $(C - A)/A$. While we previously adopted certain numerical values for these quantities in order to calculate the magnitude of the

*) l. c. tome 2, §192, eqns. (m) and (n). In addition, we have suppressed two of the Tisserand terms that have found no explanation in the preceding.

**) For example, Th. Oppolzer, Bahnbestimmung der Kometen und Planeten, Leipzig 1870 and 1882, Bd. I, erster Teil.[228]

precessional velocity and the amplitude of the nutation, the situation is, in reality, that the most reliable numerical values of these ratios follow just from the observation of the precession and the nutation. Thus it is not possible, in a logical manner, to calculate in advance the precession and the nutation. In addition, there is still the fundamental physical assumption that one may regard the Earth as a rigid body for the effects computed here, an assumption to which we will return in the following part of this chapter.

We saw above that the two quantities $(C-A)/C$ and M/m_2 enter into the theoretical expression for the precessional velocity ν (equation (6′) on page 641) as well as the expression for the nutation amplitudes a and b (equation (11) on page 660). If we therefore take the two most observationally precise values—for example, ν and a—then we have two equations for the determination of the two unknowns $(C-A)/C$ and M/m_2. One finds, in such a manner, that the currently most trustworthy values of these two unknowns are*)

$$\frac{C-A}{C} = \frac{1}{304{,}9}, \quad \frac{M}{m_2} = 81{,}58.$$

These correspond to the abbreviated numerical values $1/305$ and 82 that were used above. There follows for the so-called ellipticity, with the same approximation, $(C-A)/A = 1/304$.

Moreover, numerical results obtained by other means (for example, from geodetic measurements of the Earth and from the mutual perturbations of the orbits of the Earth and Moon) agree with the given numbers, in so far as one can expect considering the greater uncertainty of the latter means of determination.[230]

B. Geophysical Applications.

§5. The Euler period of the pole oscillations; theoretical treatment.

It is well known from the preceding that the *pure precessional motion* of the top under the influence of gravity represents an exceptional case, and that the motion will generally be overlaid with a periodic oscillation of the figure axis. This oscillation, which becomes imperceptibly small

*) cf. Newcomb, Fundamental Constants of Astronomy. Washington 1895, p. 133.[229]

for a sufficiently strong eigenimpulse, was designated simply as a *nutation.* We will now designate it, in distinction to the nutation discussed in the previous part of this chapter, as a *free nutation.* Our "*pseudoregular precession*" arises from the composition of a uniform precession and this free nutation.

The question now presses upon us: is the precession of the Earth's axis that was calculated in the preceding part of this chapter accompanied by an oscillation that is not caused by external forces, but rather represents a free oscillation of the system? Or, more concisely, *is the rotational motion of the Earth*, disregarding all forced oscillations, *a regular precession or a pseudoregular precession?*

The answer to this question requires the collaboration of theory and observation. We first give the theory.

The words "axis of the Earth" are ambiguous. One signifies by these words, on the one hand, the *figure axis* of the Earth—that is, the principal axis of inertia of the Earth that approximately coincides with the line that connects the north and south poles, and is therefore an axis that is *fixed in the body of the Earth*—and, on the other other hand, the *instantaneous rotation axis* of the Earth, and therefore a line that precisely connects the instantaneous north and south poles and is thus instantaneously fixed *in space.* The noncoincidence of the two meanings is directly the subject of the following discussion, in which we must distinguish well between the figure axis and the rotation axis.

The motion of the *figure axis* for pseudoregular precession was discussed on page 291. It was determined in terms of the angles ϑ and ψ by the approximate equations (see page 303, equations (11))

$$(1)\qquad \begin{cases} \cos\vartheta = \cos\vartheta_0 + a\sin\vartheta_0\,\sin\dfrac{N}{A}t, \\ \quad\psi = \dfrac{P}{N}t + \dfrac{a}{\sin\vartheta_0}\,\cos\dfrac{N}{A}t, \end{cases}$$

where a is expressed in terms of the quantity n' defined on page 296 as

$$(1')\qquad a = \frac{n'}{N\sin\vartheta_0} - \frac{AP}{N^2}\,\sin\vartheta_0.$$

The first terms on the right-hand sides of (1) give the precessional component of the motion, and do not come into consideration in the following. We only remark that the quantity P, which for the top was equal to MgE, is to be replaced by $P\cos\vartheta_0$, where P is determined by the expression (3) of page 640. The second terms give the free nutation, and interest us here exclusively. They represent a *circular*

oscillation (cf. page 305); that is, a motion in which the intersection of the figure axis with the firmament describes, if one disregards the precessional motion and the forced oscillations that are considered in the previous section, a small circle on the heavens. The apparent magnitude of the radius is a, and depends on the initial position of the impulse, to which the quantity n' in equation $(1')$ refers. The angle ϑ_0 signifies the mean inclination of the figure axis with respect to the normal to the ecliptic during this circular oscillation. The oscillation period τ—that is, the time in which the circle is traversed once—is determined by the equation

$$\frac{2\pi}{\tau} = \frac{N}{A} = \frac{C}{A}r,$$

where r, the angular velocity of the rotation of the Earth, is equal to 2π divided by the length of the sidereal day. If we take the latter as the unit of time, then $r = 2\pi$ and $\tau = A/C$. Since C is only slightly greater than A, *the oscillation period is slightly smaller than one sidereal day.*

This result was to be predicted. If, namely, the figure axis does not coincide with the rotation axis, then the former will rotate about the latter in a circular cone. If the rotation axis were to stand completely still, then the period would amount to exactly one day; if the position of the rotation axis changes slowly, then the period deviates only slightly from one day.

The approximately one-day oscillation of the figure axis may not be verified by observation, however, since the observation of the heavens necessarily refers to the change of the rotation axis. To the latter we now turn.

We must distinguish between the *change of the rotation axis with respect to space* and its *change with respect to the body of the Earth.* The former is determined by the components π, κ, ϱ of the rotation vector, and the latter by the components p, q, r; the two sets of components are related to the Euler angles φ, ψ, ϑ in equations (7) and (8) of page 45. The components π, κ, ϱ are the coordinates of a point on the *herpolhode*, and the components p, q, r are the coordinates of a point on the *polhode*.

The values of π, κ, ϱ are

$$(2) \qquad \begin{cases} \pi = \vartheta' \cos\psi + \varphi' \sin\vartheta \sin\psi, \\ \kappa = \vartheta' \sin\psi - \varphi' \sin\vartheta \cos\psi, \\ \varrho = \psi' + \cos\vartheta\, \varphi'; \end{cases}$$

they are referred to the fixed spatial coordinate system x, y, z whose

third axis coincides, in our case, with the normal to the ecliptic (since we measure the angle ϑ from this axis), and whose first axis is the ray $\psi = 0$ lying in the plane of the ecliptic (according to the general stipulation for the measurement of the angle ψ). It is more convenient, however, to use a coordinate system whose third axis coincides with the mean position of the figure axis, and that is therefore inclined by the angle ϑ_0 with respect to the ecliptic. The first axis of the new system may coincide with first axis of the old. If we denote the components of the rotation vector in this new system by π_1, κ_1, ϱ_1, then, evidently,

$$\begin{aligned} \pi_1 &= \pi, \\ \kappa_1 &= \quad \kappa \cos\vartheta_0 + \varrho \sin\vartheta_0, \\ \varrho_1 &= -\ \kappa \sin\vartheta_0 + \varrho \cos\vartheta_0. \end{aligned}$$

If we substitute from (2), there follow

$$(3)\ \begin{cases} \pi_1 = \vartheta' \cos\psi + \varphi' \sin\vartheta \sin\psi, \\ \kappa_1 = \quad \vartheta' \cos\vartheta_0 \sin\psi - \varphi'(\sin\vartheta \cos\vartheta_0 \cos\psi - \sin\vartheta_0 \cos\vartheta) + \psi' \sin\vartheta_0, \\ \varrho_1 = -\vartheta' \sin\vartheta_0 \sin\psi + \varphi'(\sin\vartheta_0 \sin\vartheta \cos\psi + \cos\vartheta_0 \cos\vartheta) + \psi' \cos\vartheta_0. \end{cases}$$

Now it is to be considered, according to (1), that $\vartheta - \vartheta_0$, ψ, and the differential quotients ϑ' and ψ' are small quantities; if we may disregard, in addition, the precession term Pt/N, which is not of interest to us here, then all those quantities will become of the order of the nutation amplitude a. If we expand $\cos\vartheta$, then we can write, instead of (1),

$$(4)\quad \begin{cases} \vartheta - \vartheta_0 \ = -\ a \sin\left(\dfrac{N}{A}t\right), & \vartheta' = -\ \dfrac{aN}{A} \cos\left(\dfrac{N}{A}t\right), \\ \sin\vartheta_0\,\psi = \quad a \cos\left(\dfrac{N}{A}t\right), & \sin\vartheta_0\ \psi' = -\ \dfrac{aN}{A} \sin\left(\dfrac{N}{A}t\right). \end{cases}$$

In equations (3), only the terms of the lowest order in the small quantities should be retained. We thus set $\cos\psi = 1$, $\sin\psi = \psi$, $\sin\vartheta \sin\psi = \sin\vartheta_0 \cdot \psi$, $\vartheta' \sin\psi = 0$, etc., and obtain

$$\begin{aligned} \pi_1 &= \vartheta' + \varphi' \sin\vartheta_0 \cdot \psi, \\ \kappa_1 &= -\,\varphi'(\vartheta - \vartheta_0) + \psi' \sin\vartheta_0, \\ \varrho_1 &= \varphi' + \psi' \cos\vartheta_0. \end{aligned}$$

We further note that $\varphi' + \cos\vartheta_0\,\psi'$ is equal, according to equation (7) of page 45, to the angular velocity r of the rotation of the Earth, and therefore equal to 2π if we again choose the sidereal day as the unit of time. The last equation is then $\varrho_1 = 2\pi$; in the two first equations we may directly take $\varphi' = 2\pi$, since φ' is multiplied in these two equations

by the small quantities ψ and $\vartheta - \vartheta_0$. Thus

$$\begin{aligned}\pi_1 &= \vartheta' + 2\pi \sin\vartheta_0\, \psi,\\ \kappa_1 &= -2\pi(\vartheta - \vartheta_0) + \psi' \sin\vartheta_0,\\ \varrho_1 &= 2\pi.\end{aligned}$$

If we now insert the values of ψ, ψ', etc. from (4) and consider that $N = Cr = 2\pi C$, we finally obtain the *representation of the herpolhode* as

$$(5) \qquad \begin{cases} \pi_1 = -2\pi a\, \dfrac{C-A}{A} \cos 2\pi \dfrac{C}{A} t, \\ \kappa_1 = -2\pi a\, \dfrac{C-A}{A} \sin 2\pi \dfrac{C}{A} t, \\ \varrho_1 = 2\pi. \end{cases}$$

We thus recognize that *the rotation axis describes a circular cone in space about the direction of our third coordinate axis ϱ; that is, about the mean position of the figure axis. The period with which the rotation axis once traverses this cone is again $\tau = A/C$, and is therefore slightly smaller than a sidereal day.*

We can also say that the intersection point of the rotation axis with the firmament traverses a circle in this same time. The apparent radius of this circle, measured by the angle under which it is seen from the Earth, is (replacing the trigonometric tangent by the arc)

$$\frac{\sqrt{\pi_1^2 + \kappa_1^2}}{\varrho_1} = a\, \frac{C-A}{A}.$$

This radius is considerably smaller than the apparent radius a of the circle that the intersection point of the figure axis describes on the firmament. We found, namely (cf. page 663),

$$(6) \qquad \frac{C}{C-A} = 305, \quad \text{and therefore} \quad \frac{A}{C-A} = 304.$$

The oscillation of the rotation axis in space hardly amounts to the 300th part of that of the figure axis. Since it follows from observations, as we will see, that the angular magnitude a lies firmly on the boundary of detectability, the angular magnitude $a\,\dfrac{C-A}{A} = a/304$ will completely escape observation. *One may therefore assume, for all practical purposes, that the rotation axis stands perfectly still in space.*

The above representation of the herpolhode curve is naturally not entirely exact, since we firstly omitted the higher-order terms, and sec-

ondly neglected the precession term. Had we considered the latter, then we would have obtained, instead of a circle on the firmament, a very slightly looped cycloid.

More interesting is the study of the *polhode.* Its coordinates p, q, r are given by equation (7) of page 45 as

$$\begin{aligned} p &= \quad \vartheta' \cos\varphi + \psi' \sin\vartheta \sin\varphi, \\ q &= -\vartheta' \sin\varphi + \psi' \sin\vartheta \cos\varphi, \\ r &= \varphi' + \cos\vartheta\, \psi'. \end{aligned}$$

The last coordinate is constant; it is equal, for our choice of the unit of time, to 2π. In the first two equations, we insert for ϑ' and ψ' the values from (4), and write, with the neglect of small quantities of higher order, $\sin\vartheta = \sin\vartheta_0$, $\varphi = 2\pi t$, $N = 2\pi C$; we obtain

$$\begin{aligned} p &= -\,2\pi a\frac{C}{A}\left(\cos 2\pi\frac{C}{A}t\ \cos 2\pi t + \sin 2\pi\frac{C}{A}t\ \sin 2\pi t\right), \\ q &= \quad 2\pi a\frac{C}{A}\left(\cos 2\pi\frac{C}{A}t\ \sin 2\pi t - \sin 2\pi\frac{C}{A}t\ \cos 2\pi t\right), \\ r &= 2\pi, \end{aligned}$$

or

$$\text{(7)} \qquad \begin{cases} p = -2\pi a\dfrac{C}{A}\cos 2\pi\dfrac{C-A}{A}t, \\ q = -2\pi a\dfrac{C}{A}\sin 2\pi\dfrac{C-A}{A}t, \\ r = 2\pi. \end{cases}$$

This is the desired *representation of the polhode.* It shows us *that the rotation axis also describes a circular cone in the body of the Earth. The angle at the peak of the cone* between the figure axis and the generators of the cone is (replacing the trigonometric tangent by the arc)

$$\frac{\sqrt{p^2+q^2}}{r} = a\,\frac{C}{A}.$$

This angle is therefore $C/(C-A) = 305$ times the corresponding angle of the herpolhode cone. *The time in which the rotation axis traverses the polhode cone once* amounts to $A/(C-A) = 304$ sidereal days, or about 10 months. This time is called the *Euler period* or the *Euler cycle*, since Euler*) first gave the necessary theoretical preliminaries for the calculation of this period.

*) Mechanica sive motus scientia. Petersburg 1736, third Part, Ch. XVI, §839 ff. Theoria motus corporum solidorum seu rigidorum, Greifswald 1765, Ch. XII, §§711, 717–732. The numerical value 304, however, appears not to be present in Euler.

Equations (7) are naturally not entirely complete, since we have disregarded the precession terms in their derivation; if we would consider the latter, then certain easily assignable terms of very small magnitude and of a period equal to one sidereal day would be added to the values of p and q given above.

Moreover, the above values of p and q can also be taken immediately from the Euler equations, if one bears in mind that the motion in question is a free nutation, and correspondingly prescinds the external forces (the Sun and Moon attractions) in the calculation. For $A = B$ and $r = \text{const.} = 2\pi$, the Euler equations are (cf. page 140)

$$A\,\frac{dp}{dt} = 2\pi(A - C)q, \quad A\,\frac{dq}{dt} = 2\pi(C - A)p,$$

which give, after integration (cf. page 151, equation (6′)),

$$p + iq = ce^{2\pi i\,\frac{C-A}{A}\,t}.$$

One need only resolve this expression into real and imaginary parts in order to essentially recover (namely, up to the changed notation of the constants of integration) equations (7).

It is useful to illustrate the free nutation with the figures of the polhode and herpolhode cones in the sense of P o i n s o t, and to compare with the analogous figures that represent the forced (by the Sun and Moon attractions) precession of the axis of the Earth. This is done in Figs. 100a and 100b.

In Fig. 100a (forced precession), the motion of the axis of the Earth occurs about the normal N to the ecliptic in the repeatedly named approximate time period of 26 000 years. The angle at the apex of the herpolhode cone (actually the angle between the normal N and the *rotation axis*, which we can also take without appreciable error, however, as the angle between the normal N and the *figure axis*) is $23\,{}^1\!/\!{}_2\,^\circ$. The opening angle of the polhode cone was calculated on page 49, and is equal, according to equation (2) of that page, to $\sin 23\,{}^1\!/\!{}_2\,^\circ / \,365 \cdot 26\,000$ = approximately 0,01″; the smallness of the polhode cone was illustrated in the cited place by the statement that it intersects the surface of the Earth in a circle that is described about the north pole with a radius of only 27 cm. We therefore have a *rather broad herpolhode cone and an extremely acute polhode cone*. We could naturally not express the actual quantitative

ratio of the two cones in Fig. 100a; the polhode cone is drawn proportionally almost 10^6 times too broad. We must imagine that the polhode cone, fixed in the Earth and taking part in the Earth's rotation, rotates once per day in the *counterclockwise* sense as seen from F, and rolls without sliding on the interior of the herpolhode cone. Because of its ex-

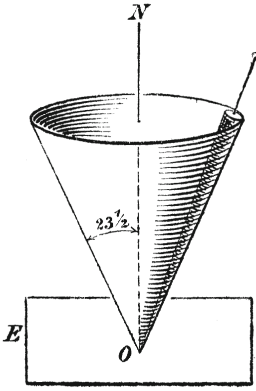


Fig. 100 a.

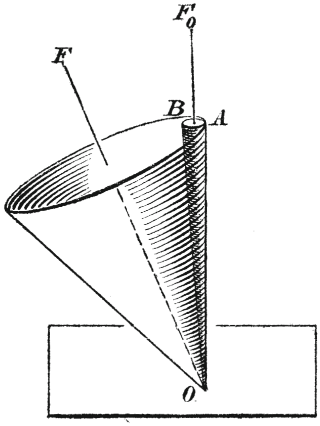


Fig. 100 b.

traordinary smallness, the polhode cone traverses the shell of the herpolhode cone only once in 26 000 years. The sense of the rolling follows from the rotational sense of the polhode cone; the rolling progresses in the figure across the front from right to left, and therefore in the *clockwise* sense as seen from N.

We now consider Fig. 100b (free nutation). The motion here occurs about the mean position of the figure axis (the vertically drawn line OF_0, as opposed to the instantaneous position of the figure axis OF). The angle at the apex of the polhode cone, according to the preceding, is equal to $a\,\dfrac{C}{A}$, and the angle at the apex of the herpolhode cone is $a\,\dfrac{C-A}{A}$; the ratio of these two angles was found to be 305. *The herpolhode cone is now considerably more pointed than the polhode cone*; the numerical proportions of the two cones in this figure could also not be expressed correctly, and the herpolhode cone must be drawn proportionally as much too obtuse. According to our formulas, the absolute magnitudes of the cone opening angles depend on the quantity a, about which only observations can give information. We are therefore uncertain, for the time being, about the actual form of the polhode and herpolhode cones, and have thus given the polhode cone in Fig. 100b somewhat the same size as the herpolhode cone in Fig. 100a. Since

observations give an extremely small value of a, the polhode cone is actually extremely acute, and the herpolhode cone is correspondingly 300 times still more acute. Fig. 100b can thus give only a rough qualitative illustration of the proportions. We must now imagine that the relatively broad polhode cone that surrounds the thin herpolhode cone rotates with the angular velocity of the Earth, and rolls without sliding on the herpolhode cone. The rotational sense of the polhode cone is again *counterclockwise* as seen from F. It follows that the rolling seen from F_0 likewise follows in the *counterclockwise* sense. The contact line of the two cones represents the position of the rotation axis in space as well as in the Earth. *The rotation axis thus circulates in space in somewhat less than a sidereal day.* If, namely, the contact line returns to its original position on the herpolhode cone (OA in the figure) after one traversal of the herpolhode cone, it coincides with the generator OB of the polhode cone that we obtain if make the arc AB on the polhode cone equal to the circumference of the herpolhode cone at the distance OA from O. The ray OA, considered as a generator of the polhode cone, is consequently not returned to its original position; the time duration of the circuit of the rotation axis on the herpolhode cone will thus be somewhat smaller than the time duration in which a ray of the polhode cone circulates once, which is equal, in turn, to a sidereal day. *On the polhode cone, on the other hand, the rotation axis circulates much more slowly.* Since, namely, it advances during a sidereal day little more than the small segment AB on the polhode cone in the sense of the Earth's rotation, the traversal of the rotation axis through the entire circumference of the polhode cone lasts a considerable number of sidereal days. This number was designated above as the Euler cycle, and was found to be 304. According to the figure, and in conformity with the calculation above, the ratio between the circulation time of the rotation axis in the Earth and the circulation time in space is equal to the ratio of the circumference of the polhode cone to that of the herpolhode cone, measured at an equal distance from the apices of the cones.

In our calculations as well as in our drawings, we have with good reason separated the treatment of the forced precession from that of the free nutation, and have left the forced nutation (which one could likewise accompany with the P o i n s o t representation) entirely to the side.

In reality, a superposition of these different motions naturally occurs, and thus a superposition of the formulas, and, in a certain sense, a superposition of the figures. Unfortunately, the Poinsot representation of the rolling cones loses its primary merit of immediate clarity for such a composite motion. If we would represent, namely, the precession and the nutation in *one* figure by means of *one* pair of rolling cones, then we must supply the herpolhode cone with extraordinarily small undulations that engage corresponding undulations of the polhode cone. For the intuitive understanding of the process, however, nothing would thus be achieved.

In the interest of the following discussion, we finally go over from the now well-known polhode cone for free nutation to the circle in which the polhode cone intersects the surface of the Earth. We distinguish the intersection of the rotation axis with the surface of the Earth, the "instantaneous Earth-pole," from the intersection of the figure axis with the surface of the Earth, the "geometric Earth-pole." Our circle is the geometric locus of the instantaneous pole, and its midpoint coincides with the geometric pole. According to the preceding theory, we must expect that the instantaneous pole encircles the geometric pole in the sense of the rotation of the Earth with the period of the Euler cycle, and therefore in about 10 months. The radius of the circle as seen from the midpoint of the Earth is, according to the preceding, $a\,\dfrac{C}{A}$.

We will report in the following section on how such a motion of the instantaneous pole is made perceptible in observations of the pole oscillations. If we estimate here the chance of observability, we see that this is now much more favorable than the previous chance of verifying the spatial motion of the rotation axis. The period and the magnitude of the motion of the instantaneous pole are about 300 times as large as the period and magnitude of the motion of the intersection of the rotation axis on the firmament. Although, as we remarked, the previous motion was imperceptible, the present motion need not be.

§6. Observational verification of the pole oscillations; the Chandler period.

The possible pole oscillations that are demonstrated in the preceding section are betrayed in observations by a variability in the *latitude* of the

observation location. It is equally valid, for this purpose, whether one defines the latitude as geographic (the complement of the angle that the plumb line of the observation location forms with the rotation axis of the Earth) or as geocentric (the complement of the angle that the line connecting the observation location and the midpoint of the Earth encloses with the rotation axis). In both cases, it is a matter of an angle between a line that is fixed in the Earth and the rotation axis that is variable in the Earth. According to whether the latter approaches or recedes from the observation location, the latitude of the location will decrease or increase.

In fact, a latitude oscillation that could not be explained by observational errors has been repeatedly conjectured previously by Peters (1842) and Nyrén (1871) at the Pulkovo observatory, and by Clerk Maxwell at the Greenwich observatory in the years 1851 to 1854.[231] The amplitude of the oscillation was in the tenths of a second, and the results for the period were inconsistent. The existence of a latitude oscillation was elevated to a certainty for the first time by the particularly precise observations of Küstner at the Berlin observatory in the year 1885.[232] We will return below to the very detailed work in which Chandler*) subjected the complete presently available observational data to a thorough discussion.[233]

New light was thrown on the entire question in the year 1891, when an astronomical expedition was dispatched to Honolulu for the purpose of latitude measurements that would be compared with simultaneous observations in Berlin.[234] Honolulu lies approximately on the meridian opposite (171° to the west) to Berlin. If the latitude oscillation actually has its basis in the change of the rotation axis of the Earth, then they must be expressed in the opposite sense at the two stations (cf. Fig. 101). The latitude in Berlin must increase if it decreases in Honolulu; a maximum of the latitude in Berlin must coincide with a minimum in Honolulu, etc. How completely this expectation is confirmed is shown in the following two

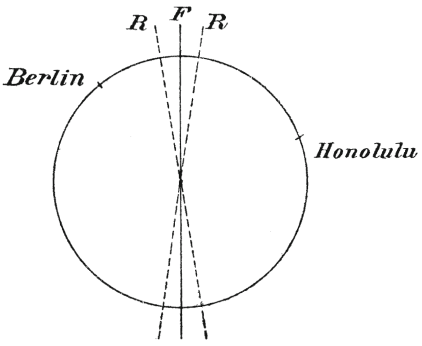


Fig. 101.

*) Astronomical Journal Vol. XI, XII, XV, XIX, XXI, XXII (1891–1902).

diagrams (Fig. 102).*) The abscissa of the figures signifies the time during the years 1891 and 1892, and the ordinate gives the deviation of the geographic latitude from its mean value, in a measure that is evident from the written numbers. The amplitude of the oscillation is,

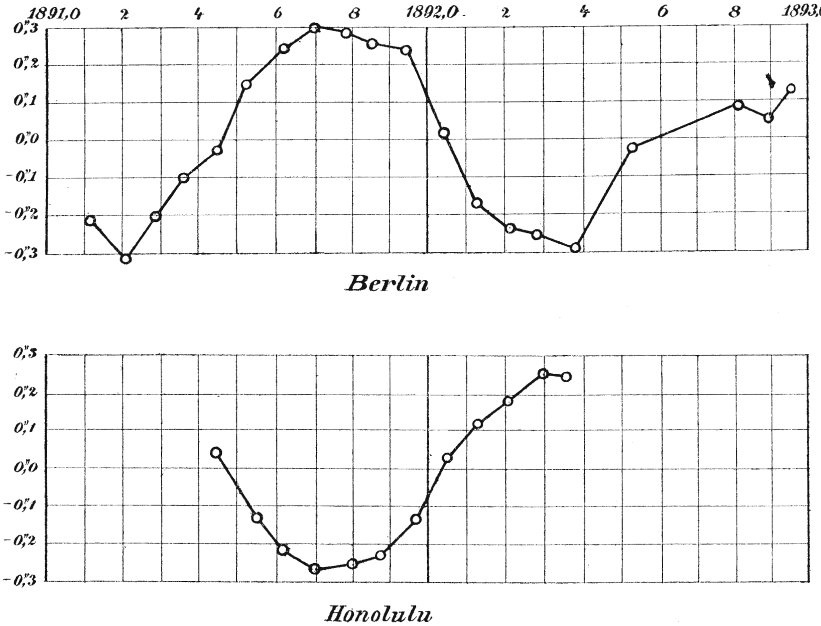


Fig. 102.

as we see, approximately equal for the two stations; it lies between $0'',2$ and $0'',3$. But above all we see that *the phase for the two stations is precisely opposite.* It is most obviously substantiated through the latter fact *that the latitude oscillation has its basis in the change of position of the rotation axis, and that this axis therefore executes a certain motion relative to the Earth.*

Two stations lying on opposite meridians, such as Berlin and Honolulu, obviously give only one component of the motion, the component with respect to the meridional plane through the two stations. For a complete knowledge of the motion, in contrast, two stations whose meridians form an approximate right angle will suffice. If more stations

*) We take these figures from the Proceedings of the 1895 Berlin Conference of the *Internationale Erdmessung*, Berlin 1896, plate 4.

are available, in particular stations that lie on opposite meridians, then their results can be confirmed with respect to each other.

Fig. 103 represents the positions of the observation locations that are the basis for the determination of the pole oscillations that was initiated by the Permanent Commission of the International Geodetic Association.[235] The majority of the European stations lie at an approximate right angle, as seen from the North Pole, to the principal American stations. The collected observational material will be processed by Th. A l b r e c h t[236] in Potsdam, and continuously released from the Central Bureau of the International Geodetic Association. We take Fig. 104, which summarizes the observational results from 1890 to 1900, from the latest report.*)

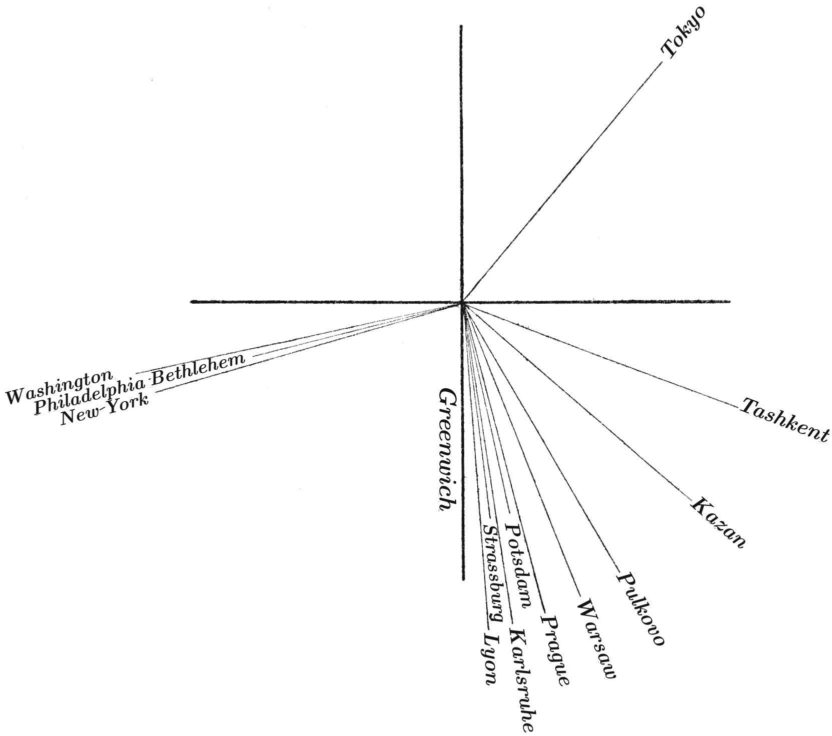


Fig. 103.

This figure represents the path of the pole in the named time interval; for clarity of the drawing, the dotted line corresponds to the first five

*) Berlin 1900. Earlier communications are found in the proceedings of the named commission from the 1896 Conference in Lausanne. Cf. also the Astron. Nachr. Nr. 3808 for the years after 1900.

years, and the solid line to the last five years. The inscribed numbers signify the dates (year and tenths of a year) for which the observations at the collected stations were reduced. The mean error for a single pair of coordinates of the instantaneous pole is given as $0'',03$. This relatively small mean error is achieved, however, only by deriving each coordinate from a large number of individual observations, which themselves have a much larger mean error. The origin of the coordinate system corresponds to the mean position of the instantaneous pole, or, as we can also say, to the geometric pole.

Motion of the North pole of the Earth's axis.

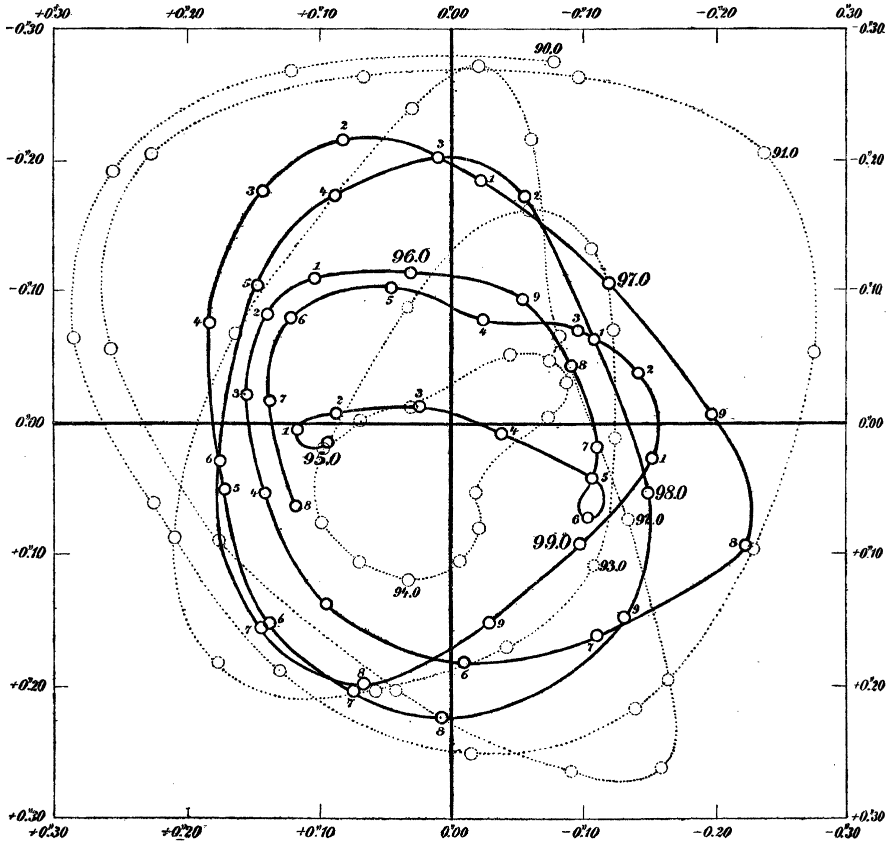


Fig. 104.

We now compare this figure with the preceding theory of the pole oscillations.

It is apparent at first glance that the pole trajectory satisfies no simple mathematical law with precision, but has an apparently accidental character and is greatly contorted. The simple relations of celestial mechanics are evidently no longer decisive for the present problem; we find ourselves here, rather, in the complicated domain of geophysics. According to the abstract theory of the preceding section, the trajectory should be a *circle*; this is, in reality, out of the question; only at the beginning of the observational time interval is a circular form somewhat approximated. In fact, we will soon recognize a series of incalculable perturbations that influence the pole motion and remove it from theoretical regularity.

On the contrary, it is to be emphasized that the *sense of the pole motion* thoroughly coincides with the theoretically required sense of the Earth rotation, if we remove a temporary irregularity in the interval from 95,0 to 95,6. In this range, either the temporary perturbations that will be discussed later have occurred to such a degree that the pole trajectory in the figure is drawn over the coordinate origin by these perturbations, or the loop is attributable to observational error, which is by no means to be excluded, since a correction in the coordinates of about the given mean error suffices to remove the entire irregularity.

For what concerns the *amplitude of the pole oscillation* (that is, the radius of the pole trajectory), this amounts in degree measure to a maximum of about $1/4''$ and in the mean to perhaps $1/8''$. The undetermined quantity a in the formulas of the preceding section is thus to be set equal, in the mean, to approximately $1/8''$. The mean distance e from the geometric pole to the instantaneous pole on the surface of the Earth follows as approximately

$$e = a \times \text{Earth radius} = \frac{\pi}{180 \cdot 60 \cdot 60} \frac{1}{8} \cdot \frac{2}{\pi} 10^7 = \text{circa } 4 \text{ m.}$$

In the years 1890 to 1895, this mean distance decreased on the average, and from 1895 to 1900 it increased; it then became smaller, but now (1902) has already gone over again into an increase, as emerges from the supplementary publication to our figure in the Astron. Nachr. (cf. the preceding footnote).

The primary interest, however, is concentrated on the question of the *period of the pole motion*. Here an initially astonishing departure from the theory appears, which is all the more noteworthy as it appears to be thoroughly substantiated. While the theory demands a period of

approximately 10 months, an examination of Fig. 104 yields a period of about 14 months. To see this, we proceed rather roughly, but with sufficient precision for our purpose, as follows. We first imagine the obviously irregular loop from 95,0 to 95,6 extended downward, so that it gives, together with the adjacent curve segments, a counterclockwise circuit about the coordinate origin that is equivalent to the other circuits, and then count the number of circuits from 90,0 to 99,4. There are directly 8 of these circuits that surround the pole in 9,4 years. Thus the duration of one circuit, or the period of the pole oscillation, amounts to

$$\frac{9{,}4}{8} \cdot 12 = 14{,}1 \text{ months.}$$

While we expected to find the ten-month Euler period, the observations indicate an essentially longer period.

The credit for discovering the longer period that emerges here belongs to the American astronomer Chandler. In the previously cited comprehensive works, Chandler examined the collected observational data from 1840 to 1891 purely numerically, without theoretical prejudice, and was thus led to a period of 427 days = ca. 14 months, a period that is called, in contrast to the *Euler* period, the *Chandler* period.

Without first entering into the theoretical explanation of this period, we wish to provide an image, through only the discussion of the observations recorded in Fig. 104, of the extent to which the pole oscillation can be represented by the assumption of a 14-month period. We will not use the laborious and fundamental calculational procedure of Chandler, but rather an obvious graphical procedure.

Let $w = x + iy$ be the vector from the coordinate origin to the current position of the instantaneous pole. Were the motion of the pole entirely created by *one* period τ_1 (for example, 14 months), then we could simply write

$$w = a\,e^{2\pi i \frac{t}{\tau_1}} + a'\,e^{-2\pi i \frac{t}{\tau_1}}; \tag{1}$$

were the motion, moreover, a purely circular motion, then one of the two constants a and a' (say, a') would vanish; at the same time, the radius of the pole trajectory would be determined by the absolute value of the other constant a. We can, however, directly account

for the case of an elliptical pole oscillation by assuming that a and a' are, in general, nonvanishing complex quantities.

The complexity of Fig. 104 shows immediately that the representation by *one* period does not suffice. We thus make the more general assumption

$$(2)\qquad w = a e^{2\pi i \frac{t}{\tau_1}} + a' e^{-2\pi i \frac{t}{\tau_1}} + b e^{2\pi i \frac{t}{\tau_2}} + b' e^{-2\pi i \frac{t}{\tau_2}} + \cdots,$$

in that we seek to represent the actually observed motion through the superposition of two (or more) oscillatory motions. It is very easy to eliminate the known 14-month period from the pole motion. For this purpose we form, according to equation (2),

$$\begin{aligned} w_{t+\tau_1} - w_t &= b\Big(e^{2\pi i \frac{t+\tau_1}{\tau_2}} - e^{2\pi i \frac{t}{\tau_2}}\Big) + b'\Big(e^{-2\pi i \frac{t+\tau_1}{\tau_2}} - e^{-2\pi i \frac{t}{\tau_2}}\Big) + \cdots \\ &= b\Big(e^{2\pi i \frac{\tau_1}{\tau_2}} - 1\Big)e^{2\pi i \frac{t}{\tau_2}} + b'\Big(e^{-2\pi i \frac{\tau_1}{\tau_2}} - 1\Big)e^{-2\pi i \frac{t}{\tau_2}} + \cdots \\ &= c e^{2\pi i \frac{t}{\tau_2}} + c' e^{-2\pi i \frac{t}{\tau_2}} + \cdots, \end{aligned}$$

where c and c', just like the previous b and b', signify unknown complex quantities. If, therefore, a period τ_2 is included in the pole motion in addition to τ_1, then this additional period must be directly expressed in the difference $w_{t+\tau_1} - w_t$, in which the primary period τ_1 is eliminated, just as the period τ_1 is expressed in w itself.

One determines the difference $w_{t+\tau_1} - w_t$ in the simplest manner by the following *graphical construction* on the pole trajectory in Fig. 104.*) Let $\tau_1 = 14$ months $= 1{,}17$ years. One then compares, for example, the location of the pole for the time 90,0 with the location for the time 91,17. The line that connects these two locations gives us the magnitude, direction, and sense of the vector $w_{t+\tau_1} - w_t$ for $t = 90{,}0$. It is only necessary to carry over this segment with a parallel straightedge from Fig. 104 into a new figure (Fig. 105a). We thus obtain a vector emanating from the coordinate origin of this new figure, of which only the endpoint is marked, and which is denoted by the number 90,0. We proceed in a similar

*) This graphical procedure may be new and useful for many similar cases. Cf. also, for what concerns the analytic rules for finding "hidden periodicities," the report of H. Burkhardt: Entwickelungen nach oscillierenden Funktionen. Jahresbericht der deutschen Mathematiker-Vereinigung, Bd. 10 (1901), pp. 312–332. More recently, A. Schuster has given a general method (the construction of a so-called "periodgraph") through which the question may be decisively settled. Cf. Nature, Vol. 66 (1902), pp. 614–618.[237]

manner for the two locations 90,1 and 91,27 in Fig. 104, and thus obtain in Fig. 105a a point that corresponds to the difference $w_{t+\tau_1} - w_t$ for $t = 90{,}1$, and is denoted by 1. Proceeding in this manner, we derive from the pole trajectory in Fig. 104 a new pole trajectory that is free of the primary term of the motion. The so-derived pole trajectory is, as one sees, still more convoluted and irregular than the original. In order that the figure should not become too unclear, the derived pole trajectory is separated into two parts. Fig. 105a gives the time interval from 90,0 to 94,0, and Fig. 105b gives the time interval from 94,0 to 98,0. In total, therefore, the positions between 90,0 and 99,17 are used for the derived pole trajectory. For clarity, half of the

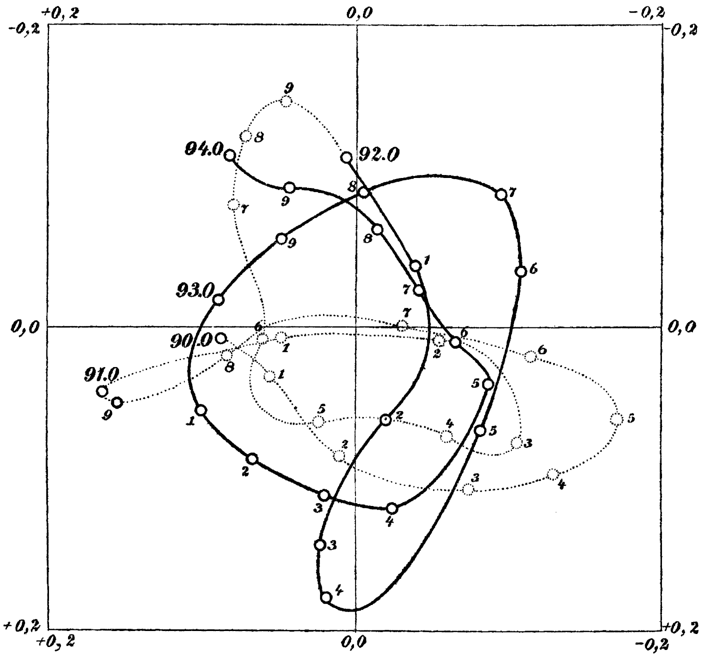


Fig. 105 a.

derived pole trajectory in each figure is dotted, and the other half solid.

A look at the derived pole trajectory shows that its dimensions are smaller than those of the original figure. While the amplitude in Fig. 104 reaches approximately $0'',30$, the amplitude in Figs. 105a and 105b exceeds $0'',15$ at only a few positions. This result is in no way self-evident, since the amplitude can just as well increase as decrease in the formation of the difference between $w_{t+\tau_1}$ and w_t.

We thus conclude that a period of 14 months is in fact present in the pole oscillation, and that approximately half of the total observed oscillation is attributable to a motion of this period.

We further conclude from the previously mentioned greater convolutedness of the new figures *that the circumstances that cause the residual amount of the pole oscillation, after the removal of the 14-month period, are less regular and less simple than those that determine the character and the period of the primary motion.* If one considers, in particular, the time interval from 96,5 to 97,5 in Fig. 105b, one

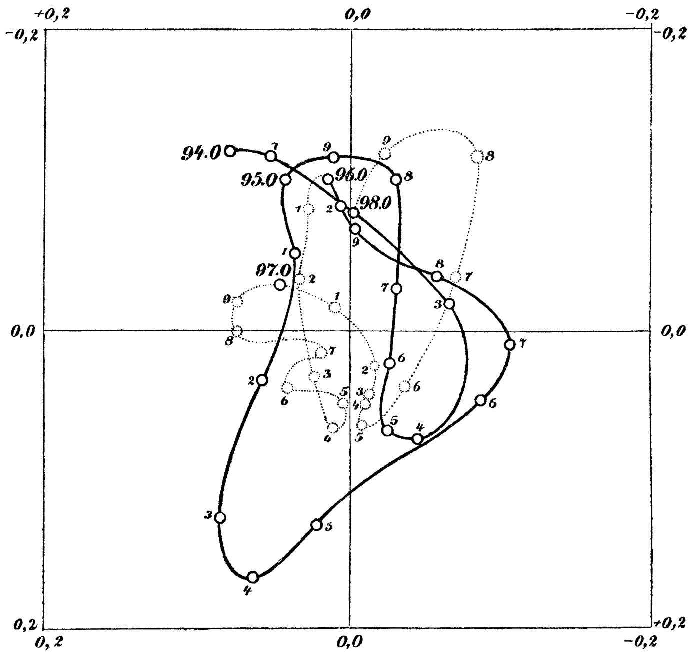


Fig. 105 b.

acquires the impression that the zigzags of this segment of the trajectory run randomly, and will entertain the suspicion that they may be completely attributable to observational error. The mean error of the pole trajectory that is derived by our construction turns out to be somewhat greater than that of the original trajectory (0″,03), so that the positions of the points of the derived pole trajectory are determined only up to about 0″,05.

A certain regularity, however, cannot be mistaken in the derived pole trajectory. In both Fig. 105a and Fig. 105b, the points denoted by

the same numbers lie, as a rule, rather close to one another. The full-year numbers, for example, are all found in the upper left quadrant of the figures, or in its immediate vicinity. To make this still more clear, one could insert in Fig. 105 the polygons of the various tenths of the years, by connecting with straight lines the positions of the pole with equal numbers 1, 2, ... in the different years. All these polygons would turn out to be relatively small. After the course of a year, the "derived pole" therefore returns, by and large, to its previous position. *A period of a full year thus appears to be present in the derived pole trajectory.*

If we seek to make the presence of the yearly period probable in the same way as above for the 14-month period, by means of counting the circuits, then we must make use in elevated measure of the arbitrariness that was applied there to pull individual loops over the coordinate origin and thus smooth the curve. We consider, for example, the solid part of the two figures from 92,0 to 96,0, which exhibit the most relatively clear elongations. We imagine the path from 92,0 to 92,4 pulled to the left across the midpoint of the figure. This path then gives, together with the segment from 92,4 to 92,8, a first circuit about the coordinate origin. We have a second rather regular circuit in the trajectory from 92,8 to 93,8. As we go over to the other figure, we imagine that the loop from 94,0 to 94,5 is pulled downward and to the left; we can then count a third circuit until 94,9. The time from 94,9 to 96,0 gives a fourth full circuit. We thus have in total, if we allow the arbitrary displacements of the trajectory as valid, directly four circuits in four years, and thus a yearly periodicity of the derived path.

This result has already been discussed by C h a n d l e r on the basis of his calculational reduction of the observations. On a similar basis, v a n d e S a n d e B a k h u y z e n later sought to represent the yearly component of the pole motion formulaically.[*])[238] He found, after subtraction of the 14-month pole oscillation, that the mean yearly pole trajectory is an ellipse whose major axis is equal to $0'',104$ and is directed toward the 19[th] meridian east of Greenwich; its minor axis amounts to $0'',044$. The ellipse runs, as is also to be seen from Figs. 105a

[*]) Akademie von Wetenschappen, Amsterdam, August 1900.

and 105b, in the direction from west to east; the pole passes the 19[th] meridian at the beginning of October.

It is natural to repeat our procedure, and to derive a new figure from Figs. 105a and 105b by elimination of the yearly period. If w now signifies the vector represented in Figs. 105 and τ_2 signifies the period of one year, then we form in a graphical manner, as above, $w_{t+\tau_2} - w_t$ and insert the result into a new figure. Only the solid part of Figs. 105 (from 92,0 to 96,0) has been adopted for further consideration, so that the resulting new Fig. 106 ranges from the marks 92,0 to 95,0, where again half the figure (from 92,0 to 93,5) is solid, and half (from 93,5 to 95,0) is dotted. Were a further period in addition to τ_1 and τ_2 contained in the pole motion, then this must become visible in our Fig. 106. It appears, however, that this is not so; *Fig. 106, rather, gives the impression that here it is a matter of individual perturbations that cause corresponding individual protrusions in the trajectory, which are followed by a retreat into the equilibrium position.* We have one such protrusion in the figure for 92,1, a second for 93,2, etc. As the comparison of Figs. 105 and 106 shows, an essential diminishment of the dimensions no longer occurs as a result of our procedure. Thus the magnitude of our figure does not illustrate the reality of the 12-month period, as it did previously for the 14-month period. *We must therefore conclude that the perturbations that are made apparent in Fig. 106 are of approximately the same order of magnitude as the influences that cause the assumed yearly circulation of the pole.* In any case, there remains a considerable remainder in the pole motion that is explained by neither a 14-month nor a 12-month oscillation. Of the 10-month Euler oscillation, there is absolutely nothing to be seen.

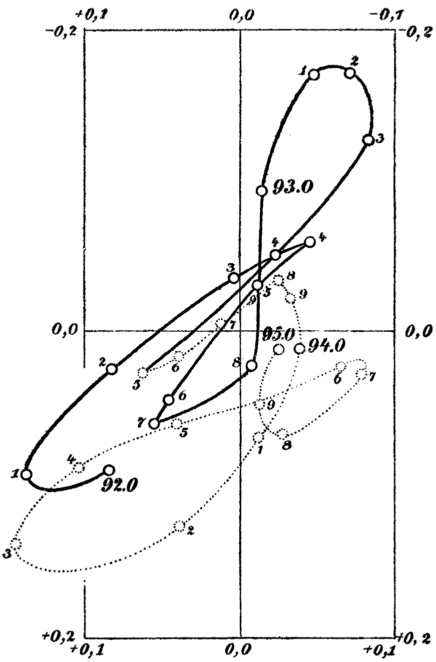


Fig. 106.

It is not impossible that the conclusions drawn here in a graphical manner are partly premature, and that they are to be modified by the further observations of the pole oscillations that are currently in progress. The last figure, in particular, must be held as rather problematic, due to the cumulative uncertainty in the repetition of our graphical process. A certain arbitrariness is also present in our derivation procedure, in so far as we need not necessarily compare the vectors w_t and $w_{t+\tau}$, but rather could just as well have taken the difference between $w_{t+2\tau}$ or $w_{t+3\tau}$ and w. The derived figures would then be different. Our results, however, coincide with the calculational results of Chandler et al.; in any case, our procedure appears to suffice for the more orienting than conclusive discussion here, and for the current precision of the relevant observations. For a conclusive judgment of the matter, a probabilistic investigation in the sense of A. Schuster (cf. the footnote on page 679) must be used to decide whether a found periodicity is to be regarded as actual or accidental.

The primary result of these considerations, the 14-month Chandler period, appears to find another certain support in the phenomenon of the tides. It it clear that the change in the location of the Earth's rotation axis must influence the motion of the ocean because of the changed centrifugal effects, and that a periodic change in position of the rotation axis must have, as a consequence, an oscillation of the mean ocean surface with the same period, assuming that the influence on the latter is sufficiently strong. Mr. van de Sande Bakhuyzen*) and Mr. Christie**) believe that this assumption is confirmed, and can demonstrate a 14-month variability of a few centimeters in the Dutch and American tidal observations.

In summary, we conclude two things from the transmitted observations. First, the pole oscillations are established without doubt, and the rotation axis of the Earth can no longer be regarded as "*der ruhende Pol in der Erscheinungen Flucht*";[240] second, the pole oscillations do not

*) Astronom. Nach. Nr. 3261.

**) Bulletin of the Phil. Soc. of Washington, 1895, Vol. XII, p. 103.[239]

have the simple regularity and the period that we would have expected according to the discussion of the preceding section.

§7. The explanation of the fourteen-month Chandler period and the elasticity of the Earth.

The old dispute over whether the interior of the Earth is molten or solid continues today in the sense of deciding whether the interior of the Earth, taken as a whole, behaves as a *solid body.* In order to avoid a mere debate over words, one must define the terms fluid and solid: a medium will be called a fluid if, under given circumstances, appreciable relative displacements of its parts can occur in its interior; it will be be called a solid if such displacements are impossible. It can remain unresolved, in the latter case, whether the deformability of the parts is caused by a kind of elastic binding (by solidity in the usual sense) or by a particularly high degree of viscosity; a fluid of sufficient viscosity (for example, asphalt) behaves perceptibly as a solid body with respect to external effects of not too long a duration, and exhibits no meaningful displacements of its parts with respect to one another. We can speak in this connection of the term *effective solidity* that is used in the English literature to denote a behavior that, under given circumstances, is analogous to a solid body of a specific degree of elasticity.

On the other hand, nothing more about its physical state should be implied by the statement that the interior of the Earth is solid. This state may differ from all that we otherwise know of fluids or solids due to the extraordinary temperatures and pressures that obtain in the interior of the Earth. Certain critical states in which the aggregate phases of matter are continuously transformed into one another may already be created in laboratory experiments; the state of the interior of the Earth, however, lies far beyond these critical boundaries. The correct standpoint is obviously not to predict the state of the interior of the Earth by risky analogies and extrapolations from laboratory experiments, but rather to deduce in reverse the average or effective state from the actual behavior of the Earth, as it is shown, for example, in the pole oscillations.

Also, it should not be claimed from the calculation of a certain elastic modulus that the entire Earth has the nature of a body with the relevant elasticity. The currently most probable and reigning view is

that the constitution of the Earth is *inhomogeneous*; that it consists, namely, of a dense and solid core (iron) and a less dense and more compliant shell (stone mantle), which are separated from one another by a not very extended layer of a viscous fluid magma (cf. the theory of E. Wiechert to be cited below[241]). We wish to include the possibility of such inhomogeneity in the word "effective" elasticity or solidity. The calculated modulus of elasticity then signifies the value of the modulus of a homogeneous elastic body that behaves just as the probably inhomogeneous Earth does with respect to the elastic effects in question.

We do not intend to enter in more detail into the discussion of the interior of the Earth, but rather emphasize only a few points of historical development.*) In the interest of the theory of volcanism, geologists have long advocated for a molten fluid interior of the Earth. The first who claimed the contrary with a scientific basis appears to be Hopkins.**) Hopkins investigated the precession and nutation of a fluid-filled spherical shell, and found that such a shell would behave quite differently from the Earth. The later and deeper investigations of Lord Kelvin***) showed that the reasoning of Hopkins was defective, and that his results are also to be corrected in essential points. Kelvin considered an oblate ellipsoidal shell instead of a spherical shell, and showed that *a completely rigid shell* would give a difference between observation and calculation in the more rapid nutations (the half-year and particularly the half-month nutations; cf. page 651) but not in the precession and the 18 $^2/_3$-year nutation,†) and that, on the contrary, all

*) For more detailed information, cf. the representation in Chap. 15 of the excellent popular scientific work of G. H. Darwin, The Tides, London 1898,[242] German edition by A. Pockels, Leipzig 1902, or the recently appearing *Kosmische Physik* by Sw. Arrhenius, Leipzig 1903.[243]

**) Researches in physical geology, Philosophical Transactions London R. Soc. 1839, 1840, 1842.[244]

***) Mathematical and Physical Papers, Vol. 3, art. 45; cf., in particular, §§21–38, summarized in Popular Lectures, Vol. 3, p. 238.[245]

†) We have confirmed a related remark of Lord Kelvin with the models of the Göttingen mathematical collection: a top whose rotary mass is replaced by a fluid-filled *oblate* ellipsoid of revolution behaves *stably* when set into rotation about its axis on a horizontal support surface, and executes a precessional motion similar to that of a solid top (on the course of the shorter nutation, the observations of our model give no clear conclusion). In contrast, a top whose rotary mass consists

these phenomena could proceed as in reality for a *somewhat compliant shell.* The astronomical facts therefore disprove only the assumption of a fluid interior in a rigid shell, an assumption that indeed is also untenable on physical grounds, since we know no material that would be fully incompliant as a thin shell. On the other hand, however, the assumption of a fluid interior in a compliant shell is refuted by the phenomenon of the tides. A thin crust of the Earth with the elastic compliability of the materials known to us would follow the deforming influence of the tidal forces almost as willingly as the water of the sea. There would then be, however, no relative motion of the water with respect to the land under the influence of these forces, but only a common rise and fall of the sea and the continents that would escape immediate perception. Thus there remains only the assumption that the Earth is, in the mean, effectively solid (solid in the sense of the prefaced explanation). This assumption is fully compatible with the existence of peripheral cavities that are full of a type of fluid magma, or also the existence of a complete circumferential fluid layer, if this fluid layer is thin in proportion to the effectively solid core and the solid crust of the Earth, so that not only the requirements of the geological theories, but also, in particular, the important results of pendulum experiments can be supported. (Cf. here too Wiechert's theory of the interior of the Earth.)

It should not be claimed at the same time that the Earth is effectively rigid. Lord Kelvin has investigated*) the degree of elastic deformability of the body of the Earth on the basis of estimating the actual height of the tides. While, as we just said, the tidal height must be reduced to zero for a primarily fluid and therefore completely compliant Earth, any finite degree of elastic deformability will account for a certain fraction of the height that must be formed by a fully rigid Earth. Kelvin estimates from this investigation that the compliability

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of a fluid-filled *prolate* ellipsoid proves to be completely *labile* under the same conditions. Since the Earth is an oblate ellipsoid, one understands that its precessional motion would not deviate essentially from that of a completely rigid body if it were filled with fluid, assuming that the crust of the Earth is, as we may assume without perceptible error in our model, absolutely rigid.

*) Thomson and Tait, Natural Philosophy II, art. 843.

of the Earth is smaller than that of glass, and approximately equal to that of steel. We will find a much sharper basis for such an estimation in the following, when we turn to the explanation of the Chandler period.

To S. Newcomb*) belongs the credit for recognizing that the period of the free nutation depends on the degree of compliability of the body of the Earth. The Euler period of 10 months corresponds to the assumption of complete rigidity; a calculation with any finite degree of elasticity gives, in contrast, a different and indeed longer period. In reverse, the Chandler period allows the assignment of a degree of elasticity for which the period of the free nutation would directly take on the observed duration of 14 months.

In the literature, these phenomena are investigated most fundamentally in a work by S. S. Hough**) that begins from the differential equations of elasticity for a rotating spheroid. We will obtain the results of Hough in a much simpler way by making use of a theorem of Chap. VII, §8 (page 607). In that section, the period of the free nutation, or, equivalently, the period of the force-free precession, is calculated for a deformable spheroidal top under the assumption that the deformation caused by the centrifugal effect of the rotation is opposed by elastic resistance only. This assumption does not apply for a body of the dimensions of the Earth, since here it is essential to consider the mutual gravitational forces among the elements.

We must therefore begin with some preliminary remarks on these gravitational forces, and on the way in which they are combined with the effect of the elastic forces. We arrange the following discussion in a series of individual problems.

First problem. A homogeneous, incompressible fluid mass stands under the influence of the mutual gravitation of its parts, and would form, in the state of rest, a sphere of radius R; it is set into rotation about a fixed axis with angular velocity ω. A possible equilibrium

*) On the dynamics of the Earth's rotation with respect to the periodic variations of Latitude, Monthly Notices Astr. Soc. London (1892), Vol. 52, p. 336,[246] and Remarks on Mr. Chandler's Law of Variation of Terrestrial Latitude, Astronomical Journal, Vols. 11, 12, 19.

**) On the Rotation of an elastic Spheroid, Philos. Transactions R. Soc. London (1896) Vol. 187, p. 319.[247]

form of the fluid is then an oblate ellipsoid of revolution that has the rotation axis as the symmetry axis (the Maclaurin ellipsoid[248]). *The ellipticity of this ellipsoid is given, under the assumption that it is small, by the formula*

$$\varepsilon_1 = \frac{5}{4}\,\frac{\omega^2 R}{g}, \tag{1}$$

where g signifies the gravitational acceleration at the surface of our fluid.

The numerical values for the Earth, expressed in meters and seconds, are

$$\omega = \frac{2\pi}{24 \cdot 60 \cdot 60}, \quad R = \frac{2}{\pi}\,10^7, \quad g = 9{,}81, \quad \frac{\omega^2 R}{g} = \frac{1}{289}, \quad \varepsilon_1 = \frac{1}{231}. \tag{2}$$

If we wish not to call upon the (very well known) formula for the potential of an ellipsoid, we can derive equation (1) advantageously by taking up our previous representation of the "Earth-ring." Just as we did previously for the rigid Earth, we now replace our ellipsoidal fluid by a sphere (radius R, mass M) and a ring (radius R, mass m) that lies in the equatorial plane of the ellipsoid. As we saw in §1 of this chapter, the gravitational potential of the combination of the ring and the sphere will be equal, up to terms of the second order inclusive, to the potential of any other mass distribution with the same principal moments of inertia. Let the moments of inertia of our fluid mass about the rotation axis (z-axis) and about two perpendicular axes (y- and x-axes) be C, A, and A. By the ellipticity we understand, as previously, the ratio*)

$$\varepsilon = \frac{C - A}{A}.$$

The mass m of the ring is to be chosen, according to equation (1) of §1, as

$$m = \frac{2(C - A)}{R^2} = \frac{2A}{R^2}\,\varepsilon = \frac{4}{5}\,M\varepsilon, \tag{3}$$

*) In addition to this definition, the definition

$$\varepsilon = \frac{a - b}{a}$$

also appears in the literature, where a is the major axis and b is the minor axis of the ellipsoid. One is easily convinced, with consideration of page 600, that for a *homogeneous* mass distribution this definition coincides, up to the higher powers of ε, with ours.

where we have introduced for A the approximate value $A = \frac{2}{5}MR^2$, which is the moment of inertia of a sphere of radius R.

If r is the distance of an arbitrary external point P from the midpoint of the fluid mass, then the potential of the sphere and ring is

$$V = f\Big(\frac{M}{r} + \frac{m}{r}\frac{1}{2\pi}\int\frac{d\varphi}{\sqrt{1+(R/r)^2-2(R/r)s}}\Big),$$

where the integration is extended over the circumference of the ring. Here s signifies, as on page 639, the abbreviation

$$s = \frac{xx' + yy' + zz'}{rR},$$

where x, y, z are the coordinates of P and x', y', z' are the coordinates of a point of the ring. If we lay the xz-plane through the point P and the xy-plane through the ring, and denote the angle between OP and OX by Θ, then

$$\begin{aligned} x &= r\cos\Theta, & y &= 0, & z &= r\sin\Theta,\\ x' &= R\cos\varphi, & y' &= R\sin\varphi, & z' &= 0, \end{aligned}$$

and thus

$$s = \cos\Theta\cos\varphi.$$

The power expansion of the square root yields, as on page 639,

$$\frac{1}{\sqrt{1+(R/r)^2-2(R/r)s}} = 1 + \frac{R}{r}s + \Big(\frac{R}{r}\Big)^2\Big(\frac{3}{2}s^2 - \frac{1}{2}\Big) + \cdots,$$

and the execution of the integration gives

$$(4)\quad \begin{cases} \dfrac{1}{2\pi}\displaystyle\int_0^{2\pi}\frac{d\varphi}{\sqrt{1+(R/r)^2-2(R/r)s}} = 1 + \Big(\dfrac{R}{r}\Big)^2\Big(\dfrac{3}{4}\cos^2\Theta - \dfrac{1}{2}\Big) + \cdots \\ \qquad\qquad = 1 + \dfrac{3}{4}\Big(\dfrac{R}{r}\Big)^2\Big(\cos^2\Theta - \dfrac{2}{3}\Big) + \cdots . \end{cases}$$

The potential of the attraction then becomes, if we insert for m the value from (3),

$$V = fM\Big(\frac{1}{r} + \frac{4}{5}\frac{\varepsilon}{r} + \frac{3}{5}\varepsilon\frac{R^2}{r^3}\Big(\cos^2\Theta - \frac{2}{3}\Big) + \cdots\Big).$$

We seek the value of V on the surface of the fluid mass, whose equation we may write as[249]

$$(5)\qquad r = R\Big(1 + \varepsilon\Big(\cos^2\Theta - \frac{2}{3}\Big)\Big)$$

(cf. equation (2) of page 601, where we have taken for the mean radius, there denoted by m, the approximate value R). Because of the smallness of ε, we can write

$$\frac{1}{r} = \frac{1}{R}\Big(1 - \varepsilon\Big(\cos^2\Theta - \frac{2}{3}\Big)\Big).$$

We insert this value into the first term of the expression for V; we may further directly set $r = R$ in the subsequent terms of the potential, which contain ε and are therefore to be considered as small. There follows

$$V = \frac{fM}{R}\Big(1 + \frac{4}{5}\,\varepsilon - \varepsilon\Big(1 - \frac{3}{5}\Big)\Big(\cos^2\Theta - \frac{2}{3}\Big) + \cdots\Big). \tag{6}$$

The surface of the rotating fluid must be a surface of constant pressure. It follows, according to the fundamental principles of hydrodynamics, that the potential energy of the forces acting on a unit mass at the surface of the fluid must also be constant. These forces are, on the one hand, gravity, and, on the other hand, the centrifugal force. The potential energy of the latter, calculated for a unit mass at an arbitrary point of the rotating ellipsoid, is

$$U = \frac{1}{2}\,\omega^2(x^2 + y^2) = \frac{1}{2}\,\omega^2 r^2 \cos^2\Theta;$$

at the surface of the ellipsoid, where r is approximately R, the potential energy U becomes, with a small formal change,

$$U = \frac{1}{3}\,\omega^2 R^2 + \frac{1}{2}\,\omega^2 R^2\Big(\cos^2\Theta - \frac{2}{3}\Big). \tag{7}$$

Thus if the sum $V + U$ on the surface of the fluid is to have a constant value—that is, a value independent of Θ—it is necessary that the factors of $\Big(\cos^2\Theta - \frac{2}{3}\Big)$ in (6) and (7) should be oppositely equal. This gives the equation

$$\frac{2}{5}\,\frac{fM}{R}\varepsilon = \frac{1}{2}\,\omega^2 R^2, \tag{8}$$

from which follows

$$\varepsilon = \frac{5}{4}\frac{\omega^2 R^2}{fM}.$$

If we introduce the gravitational acceleration g at the surface of our fluid mass—namely, $g = fM/R^2$ (approximately)—then we directly obtain the value of ε given above in equation (1).

Equation (1) was given by C l a i r a u t, and is a fundamental formula in the theory of the figure of the Earth.[250] It assumes that the density of the fluid is constant, and that the centrifugal force is held in equilibrium only by gravity.

As is well known, one designates the functions of Θ that appear as coefficients of the various powers of R/r in the expansion (4) of the reciprocal distance between the two points as *spherical functions.**) The ex-

*) More precisely said, as *spherical surface functions.* A *spatial spherical function* is any expression that is homogeneous in the rectangular coordinates x, y, z and

pression $\cos^2\Theta - 2/3$ is such a function, and indeed a "spherical function of the second order." The series (6) represents, we can say, the expansion of the potential in spherical functions. Further, we have also ordered the expression for U in equation (7) in spherical functions. In the equilibrium condition (8), finally, we have compared the two terms of V and U that contain our spherical function of the second order with one another, and thus arrived at the calculation of the ellipticity. If we denote by U_2 and V_2 the considered terms in the expansion of U and V, where we calculate V_2 with the ellipticity 1, and, with consideration that the centrifugal force represents a perturbation of a small magnitude, completely disregard the ellipticity in the calculation of U_2, then we can write the equilibrium condition schematically as

$$\varepsilon V_2 = U_2, \quad V_2 = \frac{2}{5}\frac{fM}{R}\left(\cos^2\Theta - \frac{2}{3}\right). \tag{9}$$

For a more detailed development of the theory, particularly for substantial values of the ellipticity, we must refer to the literature.*)

Second problem. A solid homogeneous elastic sphere of radius R and density ϱ rotates with the angular velocity ω about one of its diameters. It is thus transformed into an ellipsoid of revolution that has the rotation axis as its axis of symmetry. The elastic behavior of the material is determined by the modulus of elasticity (E) and the assumption that the material is incompressible, and thus that the Poisson ratio of transverse contraction to longitudinal extension has the special value $1/2$. The latter assumption simplifies the calculations, and has no significant effect on the result.

The ellipticity of the resulting ellipsoid is then

$$\varepsilon_2 = \frac{15}{38}\frac{\varrho\omega^2R^2}{E}. \tag{10}$$

In order to first give a numerical example that is related to the proportions of the Earth, we choose, in CGS units,

$$\varrho = 5{,}5, \quad \omega = \frac{2\pi}{24\cdot 60\cdot 60}, \quad R = \frac{2}{\pi}\,10^9,$$

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satisfies the potential equation $\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = 0$. A spherical surface function results from a spatial spherical function if one sets $x^2 + y^2 + z^2 = \text{const.}$ in the expression for the latter.

*) H. Lamb, Hydrodynamics, Cambridge 1895, Chap. XII, p. 580. W. Wien, Hydrodynamik, Leipzig 1900, Kap. VIII, p. 303. Thomson and Tait, Natural Philosophy II, Cambridge 1895, art. 771, 793 ff. A more detailed literature review is given by A. E. H. Love, Encykl. d. math. Wissensch. Bd. IV, Art. 16, Nr. 4.

and take E equal to the elastic modulus of steel; that is, approximately $2{,}2 \cdot 10^6$ (kg weight/cm^2) $= 2{,}2 \cdot 981 \cdot 10^9$ CGS units. Then

$$\frac{\varrho\omega^2 R^2}{E} = \frac{1}{184}, \quad \varepsilon_2 = \frac{1}{465}. \tag{11}$$

For the derivation of equation (10), we must return to the foundations of the theory of elasticity.

If u, v, w are the displacements of a point in the interior of the sphere with respect to the coordinate axes, which are chosen just as in problem 1, there first follows, because of the assumed incompressibility,

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0. \tag{12}$$

The elastic differential equations for an incompressible material that is subjected to centrifugal forces take the form

$$\begin{cases} \dfrac{E}{3}\Delta u + \dfrac{\partial p}{\partial x} + \varrho\dfrac{\partial U_2}{\partial x} = 0, \\ \dfrac{E}{3}\Delta v + \dfrac{\partial p}{\partial y} + \varrho\dfrac{\partial U_2}{\partial y} = 0, \\ \dfrac{E}{3}\Delta w + \dfrac{\partial p}{\partial z} + \varrho\dfrac{\partial U_2}{\partial z} = 0. \end{cases} \tag{13}$$

Here Δ signifies, as usual, the abbreviation for the second differential parameter and p is the isotropic pressure that changes from point to point, which is determined so that the condition (12) is satisfied. The quantity U_2 is the Θ-dependent potential

$$U_2 = \frac{1}{2}\,\omega^2 r^2 \left(\cos^2\Theta - \frac{2}{3}\right) = \frac{\omega^2}{6}(x^2 + y^2 - 2z^2) \tag{7'}$$

of the centrifugal force (see equation (7)). As one sees, U_2 is a spatial spherical function of the second order (cf. the footnote on page 691). We can disregard the term $\frac{1}{3}\,\omega^2 r^2$ that is added to U_2 in (7), since this term can change only the size and not the form of the sphere, and is completely without influence for an incompressible material.

The differential equations (12) and (13) are still to be supplemented by the condition that the surface of the sphere is a force-free surface. This condition states that the stresses with respect to all three coordinate directions must vanish on each surface element. Written in terms of the displacements u, v, w, the condition for the x-direction is

$$\left(\frac{2}{3}E\,\frac{\partial u}{\partial x} + p\right)\cos(n,x) + \frac{E}{3}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos(n,y) \\ + \frac{E}{3}\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\cos(n,z) = 0;$$

since for the sphere $\cos(n,x) : \cos(n,y) : \cos(n,z) = x : y : z$, we can write

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} + x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} = -\frac{3p}{E}x. \tag{14}$$

The conditions with respect to the y- and z-directions follow from (14) by the cyclic interchange of $(x\,y\,z)$ and $(u\,v\,w)$.

We now claim that equations (12) through (14) are satisfied, with the appropriate choice of the constants α, β, γ, by the assumption

$$\begin{cases} u = \alpha\dfrac{\partial U_2}{\partial x} + \beta r^2\dfrac{\partial U_2}{\partial x} + \gamma\dfrac{\partial}{\partial x}(r^2U_2), \\ v = \alpha\dfrac{\partial U_2}{\partial y} + \beta r^2\dfrac{\partial U_2}{\partial y} + \gamma\dfrac{\partial}{\partial y}(r^2U_2), \\ w = \alpha\dfrac{\partial U_2}{\partial z} + \beta r^2\dfrac{\partial U_2}{\partial z} + \gamma\dfrac{\partial}{\partial z}(r^2U_2), \end{cases} \tag{15}$$

which thus constitutes the complete solution of the posed problem.

In the following proof, we will make frequent use of the rules

$$\Delta U_2 = 0, \quad \left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right)U_2 = 2U_2, \quad \Delta(r^2U_2) = 14U_2,$$

$$\Delta\left(r^2\frac{\partial U_2}{\partial x}\right) = 10\,\frac{\partial U_2}{\partial x}, \text{ etc.},$$

which follow immediately from the definition and homogeneity of the spherical functions.

We first insert the assumption (15) into equation (12) and obtain

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = (4\beta + 14\gamma)U_2 = 0;$$

thus

$$\beta = -\frac{7}{2}\gamma \tag{16}$$

is to be chosen. We next go over to the first of equations (13) and calculate, according to (15) and (16),

$$\Delta u = (10\beta + 14\gamma)\frac{\partial U_2}{\partial x} = -21\gamma\frac{\partial U_2}{\partial x}.$$

Thus, according to (13),

$$\frac{\partial p}{\partial x} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial x},$$

and, correspondingly,

$$\frac{\partial p}{\partial y} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial y}, \quad \frac{\partial p}{\partial z} = (7E\gamma - \varrho)\frac{\partial U_2}{\partial z}.$$

We conclude, if we disregard the addition of a constant of integration, that

$$p = (7E\gamma - \varrho)U_2. \tag{17}$$

Finally, we must consider the surface condition (14). We first form, from (15),

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = \alpha\frac{\partial U_2}{\partial x} + 3\beta r^2\frac{\partial U_2}{\partial x} + 3\gamma r^2\frac{\partial U_2}{\partial x} + 6\gamma x U_2,$$
$$x\frac{\partial u}{\partial x} + y\frac{\partial v}{\partial x} + z\frac{\partial w}{\partial x} = \alpha\frac{\partial U_2}{\partial x} + \beta r^2\frac{\partial U_2}{\partial x} + 4\beta x U_2 + 3\gamma r^2\frac{\partial U_2}{\partial x} + 6\gamma x U_2;$$

equation (14) therefore demands, with consideration of (17), that

$$(2\alpha + 4\beta r^2 + 6\gamma r^2)\frac{\partial U_2}{\partial x} + (4\beta + 12\gamma)xU_2 = -3\Big(7\gamma - \frac{\varrho}{E}\Big)xU_2.$$

We insert for β the value from (16) and write, adjoining the corresponding equations for the y- and z-directions,

$$(2\alpha - 8\gamma r^2)\frac{\partial U_2}{\partial x} + \Big(19\gamma - \frac{3\varrho}{E}\Big)xU_2 = 0,$$
$$(2\alpha - 8\gamma r^2)\frac{\partial U_2}{\partial y} + \Big(19\gamma - \frac{3\varrho}{E}\Big)yU_2 = 0,$$
$$(2\alpha - 8\gamma r^2)\frac{\partial U_2}{\partial z} + \Big(19\gamma - \frac{3\varrho}{E}\Big)zU_2 = 0.$$

If the expression ($7'$) for U_2 is now introduced, then the first two equations become identical after the removal from each of a common factor. Our three equations are thus reduced to the two equations

$$2(2\alpha - 8\gamma r^2) + \Big(19\gamma - \frac{3\varrho}{E}\Big)(x^2 + y^2 - 2z^2) = 0,$$
$$-4(2\alpha - 8\gamma r^2) + \Big(19\gamma - \frac{3\varrho}{E}\Big)(x^2 + y^2 - 2z^2) = 0,$$

which should be satisfied at all points of the spherical surface $r = R$. This is possible only if

(18) $$2\alpha - 8\gamma R^2 = 0, \quad 19\gamma - \frac{3\varrho}{E} = 0.$$

The required values of our coefficients α, β, γ are determined by these equations and equation (16); they are

(19) $$\gamma = \frac{3}{19}\frac{\varrho}{E}, \quad \beta = -\frac{21}{38}\frac{\varrho}{E}, \quad \alpha = \frac{12}{19}\frac{\varrho}{E}R^2.$$

At the same time, the correctness of the assumption (15) is proven by this determination.

It is now easy to calculate the ellipticity of the ellipsoid that results from the deformation. We form, for this purpose, the expression for the radial displacement of a point on the surface of the sphere; namely,

$$\delta R = \frac{1}{R}(ux + vy + wz).$$

According to (15),

$$\delta R = \left(\frac{2\alpha}{R} + 2\beta R + 4\gamma R\right) U_2,$$

and therefore, because of (19),

$$\delta R = \frac{15}{19}\frac{\varrho}{E} R U_2. \tag{20}$$

On the other hand, this same displacement, if we calculate it from the ellipsoid equation (5), is

$$\delta R = r - R = R\varepsilon\left(\cos^2\Theta - \frac{2}{3}\right). \tag{21}$$

If we take the surface value of U_2 from ($7'$), then the value $\varepsilon = \varepsilon_2$ given in (10) in fact follows by comparison of (20) and (21).

We wish to represent the result of the comparison of (20) and (21) in the soon to be useful form

$$\varepsilon W_2 = U_2, \quad W_2 = \frac{19}{15}\frac{E}{\varrho}\left(\cos^2\Theta - \frac{2}{3}\right). \tag{22}$$

The quantity W_2 is a spherical function through which the elastic behavior of our sphere is characterized. Equation (22) (and similarly equation (9)) can be called a constraint equation, since it gives the oblateness of the outer surface through which the centrifugal forces will be directly canceled.

The preceding results are derived under more general assumptions by Lord Kelvin.*) Lord Kelvin considers, instead of an incompressible elastic body, a *general* elastic body, and instead of the particular spherical function U_2, a perturbation function U_n of order n; he also considers the increase of density toward the center that is verifiable for the Earth. By presuming the simplest assumptions, we succeeded in significantly shortening the Kelvin calculation.

Third problem. We again consider a solid sphere of elastic material that rotates with the angular velocity ω. In addition to the elastic resistance to a change of form, we consider the resistance produced by the mutual gravitation of the parts of the sphere. For the same elastic behavior as in the previous case, the ellipticity must now be smaller, since the resistance to the change of the spherical form is increased. *We claim that the ellipticity is now calculated from the formula***)

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*) Thomson and Tait, Natural Philosophy, Part II, particularly art. 834. Sir W. Thomson, Mathem. and Phys. Papers, Vol. III, art. 45. Cf. also A. E. H. Love, Elasticity, Cambridge 1892, Chap. X.

**) Thomson and Tait, Natural Philosophy, art. 840.

$$\frac{1}{\varepsilon_3} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}, \tag{23}$$

where ε_1 and ε_2 are determined by equations (1) and (10).

The proof is contained in equations (9) and (22). When gravitation alone or elasticity alone acted against the centrifugal forces, we found

$$\varepsilon_1 V_2 = U_2 \quad \text{or} \quad \varepsilon_2 W_2 = U_2. \tag{24}$$

If the two resistances together produce equilibrium against the centrifugal forces, the equilibrium condition is

$$\varepsilon_3 V_2 + \varepsilon_3 W_2 = U_2,$$

where ε_3 is the ellipticity that is now present.

If we divide this equation by $\varepsilon_3 U_2$ and express the ratios V_2/U_2 and W_2/U_2 in terms of ε_1 and ε_2 according to (24), we obtain exactly the formula (23) that is to be proved.

For an experimental sphere of moderate dimensions, ε_1 is extraordinarily large compared with ε_2. From (1) and (10), namely, there follows

$$\frac{\varepsilon_1}{\varepsilon_2} = \frac{19}{6}\frac{E}{\varrho R g}.$$

The quantity g, which signifies the gravitational acceleration at the surface of our test sphere, is always smaller than the gravitational acceleration of the Earth in the same proportion as the radius of the sphere is smaller than the radius of the Earth (equal mean density of the sphere and the Earth assumed). From the smallness of g as well as the bigness of E, it follows that ε_2 will be negligible compared with ε_1, and therefore that $1/\varepsilon_1$ will be negligible compared with $1/\varepsilon_2$. Equation (23) states, in this case, that $\varepsilon_3 = \varepsilon_2$; that is, that one can disregard the influence of gravity on the centrifugal deformation under laboratory conditions. With the enlargement of the dimensions of the sphere, however, $\varepsilon_1/\varepsilon_2$ decreases quadratically; for a sphere of the size of the Earth and the elasticity of steel, $1/\varepsilon_1 = 231$ is no longer to be neglected with respect to $1/\varepsilon_2 = 465$. The ellipticity of such a sphere is therefore to be calculated according to (23) and would amount to about $1/700$, which is essentially smaller than it would be if elasticity alone acted against the centrifugal effect.

One must not believe, however, that the observable ellipticity of the Earth would be determined by the common action of gravitation and elasticity in the sense discussed here, so that its calculation must follow from formula (23) with the adoption of an appropriate degree of elasticity E. We must imagine, rather, that the Earth was once, as the Sun

is now, in a molten fluid state. In this state, only gravitation could maintain equilibrium against the centrifugal action. The ellipticity must therefore have amounted to $\varepsilon_1 = 5\omega^2 R/4g$. With the gradual cooling of the Earth, rigidity then appeared, and indeed, according to Lord Kelvin's depiction of this sequence of events, by a relatively rapid process. The ellipticity of the now rigid form of the Earth essentially coincides, one may assume, with the earlier fluid form. In this form, the Earth is *stress-free* for an unchanging rotation ω. The natural state of the Earth is this oblate form; elastic forces occur only in so far as a change of this original form is caused by a change of the rotational properties or by other forces, in which case the elastic forces would act to restore this stress-free form.

It thus follows that one can extract nothing about the elastic properties of the body of the Earth directly from the currently observable ellipticity. The situation here is different from that of the previously mentioned experimental sphere, whose natural state without rotation is the spherical form, and for which elastic resistance thus appears if this spherical form is changed by centrifugal effects. As a result, the form of the rotating experimental sphere of moderate dimensions will be influenced predominantly, as we saw, by the elastic forces; the actual form of the Earth, in contrast, gives evidence only of the gravitational action for the normal rotational velocity ω.

Fourth problem. In order to take a further step toward the present properties of the Earth, we now begin with an oblate ellipsoid of ellipticity ε_1 that rotates with angular velocity ω; the ellipsoid consists of a gravitating elastic material, and is in stress-free equilibrium with this form and this motion. We ask for the ellipticity ε that it would assume *if the rotation ceased.* This ellipticity will in any case be smaller than ε_1; gravity will abet the diminishment of the ellipticity, and elasticity will resist it.

We claim that the desired ellipticity ε is expressed in terms of the previously calculated ellipticities ε_1 and ε_2 (equations (1) and (10)) *as*

$$\varepsilon = \frac{\varepsilon_1^2}{\varepsilon_1 + \varepsilon_2}. \tag{25}$$

We note, in addition, the difference between the ellipticity of the

stress-free state with rotation ω and the ellipticity of the state when the rotation has ceased. This difference is called ε'; it is, according to (25),

$$\varepsilon' = \varepsilon_1 - \varepsilon = \frac{\varepsilon_1 \varepsilon_2}{\varepsilon_1 + \varepsilon_2}. \tag{26}$$

We conduct the proof in a twofold manner.

a) In the natural state of the ellipsoid (rotation ω, ellipticity ε_1), equilibrium obtains between the centrifugal forces and the gravitational forces. For this state, according to (9),

$$\varepsilon_1 V_2 = U_2.$$

In the deformed state (rotation 0, ellipticity ε), we have, in contrast, equilibrium between gravitation and elasticity. Since the elastic forces tend to deform the body toward the stress-free state (ellipticity ε_1), the elastic effects are now to be measured by the difference in ellipticity ε', and will be given by $\varepsilon' W_2$. Equilibrium between gravitation and elasticity requires that

$$\varepsilon V_2 = \varepsilon' W_2, \quad \text{or} \quad \varepsilon V_2 = (\varepsilon_1 - \varepsilon) W_2.$$

We divide this equation by U_2 and set, according to (9) and (22), $V_2/U_2 = 1/\varepsilon_1$, $W_2/U_2 = 1/\varepsilon_2$. Then

$$\varepsilon\Big(\frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}\Big) = \frac{\varepsilon_1}{\varepsilon_2}, \tag{27}$$

which coincides with equation (25).

If we begin directly from the solution to our third problem above, then we can also reach the conclusion in the following manner.

b) In the stress-free state ε_1 of the rotating ellipsoid, we imagine that the centrifugal forces cancel the gravitational forces. In order to go over to the rotationless state ε, we must apply to our ellipsoid the centrifugal forces in the reversed (centripetal) sense, and also apply the difference of the gravitational forces with respect to the previous state in the reversed (or centrifugal) sense. The elastic forces that tend to produce the state ε_1 act in the same (centrifugal) sense. Thus the elastic forces and the difference of the gravitational forces act together against the reversed centrifugal forces in the passage from the state ε_1 to the state ε_2; that is, in the ellipticity change ε'. The resulting change ε' of the ellipticity can thus be calculated immediately according to equation (23); we obtain

$$\frac{1}{\varepsilon'} = \frac{1}{\varepsilon_1} + \frac{1}{\varepsilon_2}, \tag{28}$$

which coincides with (26) —.

After settling these four preliminary problems, we now enter into the actual subject of this section, the *explanation of the Chandler period.* We recall the result of Chap. VII, §8 concerning the nutation period*) of a deformable top with an approximately spherical form. We saw on page 607 that this period is calculated from the "original" ellipticity ε that the top would have in the rest state of zero rotation, and not from the ellipticity that refers to the spheroid in rotation, which we named $\mathsf{E} = \varepsilon + \varepsilon'$. The formula is (cf. equation (12) of page 607)

$$\frac{\text{nutation period}}{\text{rotation period}} = \frac{1}{\varepsilon}. \tag{29}$$

In the case of the Earth, the rotation period is equal to one day, and ε signifies the ellipticity that the Earth would assume for zero rotation, and is therefore to be calculated according to equation (25). The ellipticity E of the rotating top is to be identified, in the case of the Earth, with ε_1; the difference of the two ellipticities was denoted by ε' in the preceding as well as previously. The distinction is that we previously thought that this additional ellipticity ε' was caused by the elastic properties of the top alone, while it is caused for the Earth by the elastic and gravitational actions taken together in the sense of equation (28). The applicability of our previous deliberations is not affected; only the presence of a deformation was assumed in that calculation, not the circumstances under which this deformation came into being. It matters equally little that we previously regarded the top as stress-free in the rotationless state ε, while, in contrast, the Earth is stress-free in the state $\varepsilon_1 = \varepsilon + \varepsilon'$, and in the imagined rotationless state ε is subjected to elastic stresses in such a measure that it would perhaps burst. For it is indifferent in the determination of the motion of a body whether a force system is added to the body or taken from it, if this force system holds the body in equilibrium.

*) With respect to the terminology (free nutation = force-free precession), cf. the beginning of §8, page 599 and below.

According to equations (29) and (25), the period of the free nutation of the Earth, expressed in days, is equal to

$$\text{(30)} \qquad \frac{\varepsilon_1 + \varepsilon_2}{\varepsilon_1^2} = \frac{1}{\varepsilon_1}\Big(1 + \frac{\varepsilon_2}{\varepsilon_1}\Big).$$

Were the material of the Earth absolutely rigid (its modulus of elasticity infinitely large, and therefore $\varepsilon_2 = 0$), then the nutation period would be calculated from the ellipticity of the *rotating* Earth, and would equal $1/\varepsilon_1$. If the Earth, however, were elastically compliant (elastic modulus finite, $\varepsilon_2 > 0$), then the nutation period would depend on the ellipticity of the *hypothetically rotationless* Earth, and an additional term would be added to $1/\varepsilon_1$. *The elastic compliability of the Earth therefore lengthens the period of the free nutation*, and indeed in the ratio $1 + \varepsilon_2/\varepsilon_1 : 1$. If we assume, for example, that the Earth has the elastic modulus of steel, then this ratio of lengthening is

$$1 + \frac{231}{465} = 1{,}5.$$

If we set the nutation period of the absolutely rigid Earth equal to 10 months (the Euler period), then the nutation period of an Earth with the elasticity of steel has a period of 15 months. Since, on the other hand, observation has given a nutation period of 14 months (the Chandler period), and therefore a lengthening ratio with respect to the Euler period of 1,4, we conclude that

$$1 + \frac{\varepsilon_2}{\varepsilon_1} = 1{,}4$$

for the material of the Earth. We can thus extract the effective degree of elasticity of the Earth. There follows

$$\varepsilon_2 = 0{,}4\,\varepsilon_1 = \frac{0{,}4}{231} = \frac{1}{578}.$$

According to equation (10), the quantity ε_2 is inversely proportional to the elastic modulus of the relevant material, and according to equation (11), $\varepsilon_2 = 1/465$ for steel. The elastic modulus of the Earth is thus calculated as 578/465 (that is, 1,24) times the elastic modulus of steel. *We must therefore ascribe to the Earth only a very slight degree of elastic compliability in order to explain in this manner the elongation of the Euler period into the Chandler period; the Earth must be, in its mean elastic behavior, somewhat less compliant than steel, or have a somewhat higher modulus of elasticity.*

This manner of conclusion still begs a point of explanation. The given value (30) for the nutation period of the Earth's axis gives, under

the assumption of rigidity ($\varepsilon_2 = 0$), the period $1/\varepsilon_1 = 231$ days. This period is different from the Euler period, and does not even amount to 8 months. The basis for this difference naturally lies in the fact that the actual ellipticity of the Earth is different from that which we calculated by hydrodynamics under the assumption of a homogeneous mass distribution. We have obviously been guilty of a certain inconsistency in the preceding, in that we determined the enlargement ratio $1 + \varepsilon_2/\varepsilon_1$ from the *theoretical formula* (30), and, in contrast, set the nutation period for the rigid behavior of the Earth equal to the Euler period, which was taken directly from the astronomical *observations* of the precession of the Earth. This inconsistency may be justified after the fact in the following manner.

The theoretical value $\varepsilon_1 = 1/231$ takes no account of the nonuniform mass distribution in the interior of the Earth, and represents only an *upper limit* for the ellipticity of the Earth, whose mean density is empirically much larger than its surface density. In fact, the actual ellipticity of the Earth (1/304 according to astronomical observations; cf. page 663) or the actual oblateness (1/298 according to geodetic measurements) is indeed smaller than the calculated value 1/231 for a homogeneous mass distribution. It is also clear that the concept of the ellipticity itself becomes undetermined for an inhomogeneous mass distribution, in so far as the two definitions on page 689 must then lead to different numerical values. We can denote the definition given in the text as the *mass ellipticity*, and the alternative definition in the footnote as the *oblateness* or the *surface ellipticity.* Numerous investigations of Radau, Callandreau, Poincaré, and older researchers are concerned with the question of what law of density increase (assumed as continuous) toward the interior of the Earth must be supplied in order to be compatible with the empirical mass and surface ellipticities, as well as with the empirical mean Earth density.[251] We refer for this subject to the summarizing presentation of Tisserand,*) to whose report, however, is to be added that E. Wiechert**) has more recently combined the collected astronomical, geodetic, and physical data into a noteworthy theory of the interior of the Earth, to which repeated

*) Tisserand, Mécanique céleste, t. 2, chap. XIV, in particular art. 110–112.

**) E. Wiechert, Die Massenverteilung im Innern der Erde, Göttinger Nachrichtern 1897, p. 221. Cf. also G. H. Darwin, Monthly Notices of the R. Astr. Soc. London. Vol. 60 (1899) No. 2. The results of Wiechert and Darwin are compared by F. R. Helmert, Sitzungsberichte der Akademie d. Wiss. Berlin 1901, p. 328.

reference has already been made. In this theory, the density of the Earth is assumed as stepwise variable; the Earth, namely, is taken to consist of a more dense metal core and a less dense stone mantle, which are separated from one another by an intermediate viscous layer. The size and mass ratios of the core and the mantle are determined so that the mass and surface ellipticities turn out correctly, and so that the surface of the mantle corresponds exactly, and the surface of the core approximately, to stress-free hydrodynamic equilibrium. We mention this work here in order to make it understandable that through appropriate assumptions on the mass distribution, the theoretical limiting value 1/231 can actually be transformed into the observed mass ellipticity value of 1/304, as well as into the observed value of the oblateness, and that therefore, in particular, the mass ellipticity of the inhomogeneous Earth is diminished in the ratio 231/304. The nutation period corresponding to the rigid constitution, which indeed was inversely proportional to the mass-ellipticity, must be enlarged, and it is natural to assume that the nutational period corresponding to the actual elastic constitution is enlarged in just this ratio with respect to the value that it would have for a homogeneous mass distribution. This assumption lies implicitly at the basis of our above explanation of the Chandler period, in which we began not from the nutation period $1/\varepsilon_1$ that is valid for a homogeneous mass distribution, but rather from the greater Euler period of 304 days due to inhomogeneity, which we then multiplied, because of the elasticity of the Earth, by the enlargement ratio $1+\varepsilon_2/\varepsilon_1$ calculated from the theoretical values of ε_1 and ε_2. The same assumption is made, for want of another more certain basis, in the work of Hough cited above.

Thus it should not be denied that our result is originally drawn from a homogeneous ellipsoid, and that the passage to the inhomogeneous Earth is necessarily bound with some uncertainty.

This uncertainty, however, concerns only the quantitative results, and not the qualitative. It is very well possible that one may find, with consideration of the inhomogeneous density distribution in the interior of the Earth, a mean modulus of elasticity that is somewhat different from the numerical value given above. In contrast, the general result confidently remains that the period of the free nutation is increased by the elasticity of the Earth, and that for a certain degree of compliability, the Euler period goes over into the Chandler period for an arbitrary mass distribution and an arbitrary structure of the interior of the Earth.

Finally, a contribution to the full understanding of the pole oscillations (or, more precisely said, the 14-month period of the same) will be made if we carry over the depiction of the motion of a deformable top, given generally in §8 of the preceding chapter, to the proportions of the Earth.

For the normal position of the rotation axis, in which the rotation axis coincides with the polar principal inertial axis, the Earth rotates uniformly about this axis with the oblateness $1/298$. The difference of the equatorial and the polar Earth radii thus amounts to $R/298$, or approximately 21 km, where R signifies the mean radius of the Earth. Now let the rotation axis be deflected by any circumstance. The ellipticity of the Earth thus remains the same (cf. page 603), but the position of the principal inertial axes does not (cf. Fig. 90 on page 602). The mass of the Earth deflects, for a fixed form of the Earth ellipsoid, toward the side of the deflected rotation axis. The mass distribution, however, is not symmetric about the rotation axis, but rather about an axis (the instantaneous principal inertial axis) that lies between the original principal inertial axis and the instantaneous rotation axis. And indeed this axis divides the angle (δ in Fig. 90) between the original principal inertial axis and the instantaneous rotation axis, according to equation (6) of page 603, in the ratio $\varepsilon'/(\varepsilon+\varepsilon')$. Since ε' was equal to $\varepsilon_1-\varepsilon$, the named ratio can also be written as $1-\varepsilon/\varepsilon_1$. Now $1/\varepsilon$ determines the duration of the Chandler period, and $1/\varepsilon_1$ that of the Euler period. Thus

$$\frac{\varepsilon}{\varepsilon_1}=\frac{10}{14} \text{ and } 1-\frac{\varepsilon}{\varepsilon_1}=\frac{2}{7}.$$

If e is the deflection of the instantaneous rotation pole on the surface of the Earth, then the deviation of the instantaneous inertia pole is $2e/7$. On page 677, we concluded from observations that e amounts, in the mean, to 4 m; the deviation of the principal inertia pole will thus be only 1,1 m. If the instantaneous rotation pole were simply to describe a circle of radius 4 m in 14 months about the original principal inertia pole (the geometric pole), then the instantaneous inertia pole must traverse a circle of radius 1,1 m about the same mean point in the same time and in the same rotation sense.

The displacement that a point on the surface of the Earth would thus experience, and which consists in part of an elevation and in part of a descent, is extremely small. We can extract it immediately from equations (3) and (5) of pages 601 and 602. In equation (3), r signi-

fies the distance between a point on the surface of the spheroid and the midpoint of the spheroid for the normal rotation ω about the original principal inertial axis; equation (5) gives the same distance for the deflected rotation axis. The difference of the two represents the displacement of the point as a result of the deformation of the spheroid; it amounts, if we set the previously designated radius m to the mean radius of the Earth and treat the angular deviation δ as a small quantity, to

$$(3) - (5) = R\varepsilon'(\cos^2\Theta - \cos^2(\Theta+\delta)) = R\varepsilon'\delta\sin 2\Theta.$$

In Fig. 90 of page 602, this quantity is represented by the thickness of the strip between the original elliptical outline and the outline of the ellipse that is deformed and rotated by ϑ. The greatest displacement is found, according to Fig. 90 and the present formula, for $\Theta = 45°$, where $\sin 2\Theta = 1$. For this latitude we can represent the displacement, if we denote by e the quantity $R\delta$ that measures the deflection of the rotation pole on the surface of the Earth, as

$$\varepsilon' e = (\varepsilon_1 - \varepsilon)e = \varepsilon_1\Big(1 - \frac{\varepsilon}{\varepsilon_1}\Big)e = \frac{1}{304}\Big(1 - \frac{10}{14}\Big)e < e \cdot 10^{-3}.$$

Since e amounts to only 4 m, the greatest displacement of a point on the surface of the Earth will be less than 4 mm.

The smallness of this displacement implies a corresponding smallness of the plumb line oscillation that is produced by the deformation of the Earth. We determine, once from equation (3) of page 601 and once again from equation (5) of page 602, the angle that the normal to the ellipsoid forms with the line that connects the relevant location to the midpoint of the Earth. This angle is (with the interchange of the angle and the tangent),

$$\frac{1}{r}\frac{dr}{d\Theta},$$

and becomes, in the first approximation,

$$\begin{aligned}&\text{according to (3)} \cdots && -(\varepsilon+\varepsilon')\sin 2\Theta,\\ &\text{according to (5)} && -\varepsilon\sin 2\Theta - \varepsilon'\sin 2(\Theta+\delta).\end{aligned}$$

The difference of the two angles, which is equal to the change of direction of the plumb line, is thus

$$\varepsilon'(\sin 2(\Theta+\delta) - \sin 2\Theta) = 2\varepsilon'\delta\cos 2\Theta.$$

The greatest plumb-line oscillation thus occurs, in agreement with Fig. 90, for $\Theta = 0$ and $\pi/2$; that is, at the poles and at the equator. It amounts to

$$\pm 2\varepsilon'\delta.$$

Since we just found that $\varepsilon' < 10^{-3}$, the greatest plumb-line oscillation will be smaller than the 500th part of the deviation of the rotation axis. Since the latter, according to Fig. 104 of page 676, is smaller than $0'',3$, the plumb-line deviation will in any case be smaller than $0'',0006$, a magnitude that is by no means accessible to observation.

Finally, the possible influence of the water of the oceans on the duration of the nutation period may be pointed out. Were the surface of the Earth fully covered with water, then this water, since it would be free to follow the influence of the centrifugal forces, would form a symmetric ring about the instantaneous rotation axis. We would then have to distinguish, in association with Fig. 90, the surface of the fluid, which will rotate for a deflection of the rotation axis by the full angle δ with respect to its original position, and the surface of the solid core of the Earth, which will be displaced toward the side of the rotation axis by only the fractional part of δ that is determined on page 603. The more extensive deformation of the fluid surface would lengthen, in its turn, the period of the free nutation; thus a part of the deviation between the Chandler and the Euler periods must be explained by the behavior of the fluid covering, and only the remainder through the elasticity of the Earth. The compliance of the Earth would thus be still smaller, or its mean elastic modulus still greater, than was found above, where we ascribed in the calculation the entire deviation to the elasticity of the Earth. In reality, however, the Earth's surface is covered not completely by water, but only $^2/_3$ so, and the mobility of the water will be restricted in a complicated manner by the form of the continents. Thus it is hardly possible that the influence of the oceans on the nutation period of the Earth's axis can be estimated a priori in an objectionless manner. One must wait, rather, until sufficient observational data on the water motion that corresponds to the pole oscillations are at hand. It was already suggested on page 684 that the tidal flow shows a 14-month period; if the existence of the same is established with certainty and its magnitude is determined approximately, the exercise of determining the influence of this flow on the problem of the free nutation of the axis of the Earth will then arise.

§8. Pole oscillations of yearly period. Mass transport and flow friction.

With the explanation of the Chandler period, only one aspect of the problem of the pole oscillations is settled. It is to be determined further

why an additional yearly period occurs, which we read from Figs. 105a and 105b of pages 680 and 681, and also why a residuum of apparently irregular and accidental perturbations (Fig. 106 of page 683) remains after the removal of the oscillation with this yearly period. The important general question is still to be answered: why is the free nutation of the Earth's axis so complicated and irregular, while the forced nutation (cf. §3 of this chapter) conforms so rigorously to a mathematical law?

The basis for this appears to be the fact that the Earth is indeed *effectively rigid* in the sense of the preceding section, but not *actually rigid*, and that its parts can be displaced to a certain degree with respect to one another. In particular, the presence of the yearly period suggests the hypothesis that such a displacement or a "mass transport" is caused by the heat of the sun, and may therefore have a meteorological origin. For the explanation of the yearly pole oscillation, one must draw upon various meteorological influences: the yearly change in snow and ice deposits, the sea storms of yearly period and the resulting water transport, and the oscillation in the level of the atmosphere. The latter can apparently be determined most quantitatively on the basis of the well-known isobaric maps for the greater part of the Earth, and yields mass displacements of astonishingly large amounts.

We take the following information from an investigation by R. S p i t a l e r.[*]) It is well known that the air pressure in the winter is higher, in the mean, than in the summer. Thus the air pressure difference between January and July in the northern hemisphere is in the mean positive, and in the southern hemisphere is negative. The distribution of the pressure difference is naturally not uniform, but rather differs essentially according to whether the considered region is fixed land or ocean, and indeed different in the sense that the water covering noticeably offsets the air pressure oscillation. In conformity with this deliberation, the isobaric maps show that the pressure excess between January and July in the northern hemisphere is concentrated over the Asiatic land mass, while the pressure excess between July and January in the southern hemisphere is grouped in islands over the three regions of South Africa, South America, and Australia. Nothing is known with certainty about the polar regions. With respect to the magnitude of the

[*]) Die periodischen Luftmassenverschiebungen und ihr Einfluſs auf die Lagenänderung der Erdachse. Petermanns Mitteilungen, Ergänzungsheft Nr. 137 (1901).[252]

pressure difference, an analysis of the isobaric maps shows the following. In the northern hemisphere between 0° and 80° latitude, there is an air mass excess in January with respect to July that is equal to 192,5 cubic kilometers of mercury, and in the southern hemisphere between 0° and 50° latitude an air mass excess in July with respect to January of 402,2 cubic kilometers of mercury! Since a cubic kilometer of mercury has a mass of $13{,}6 \cdot 10^{12}$ kg, we are concerned here with mass differences that are already somewhat comparable to the total mass of the Earth (equal to the mean density times $4\pi R^3/3 =$ ca. $6 \cdot 10^{24}$ kg).*)

With respect to the mass transport caused by ocean currents, we refer to an estimation by J. Lamp.**)

In addition to the meteorological influences, mass transports of shorter periods occur as a result of tidal flows; aperiodic mass displacements of smaller amounts are due to earthquakes, volcanic eruptions, deposits of rivers, and the secular rising and sinking of the crust of the Earth.***)

We must now ask ourselves how such mass transports influence the motion of the Earth. We can distinguish an *indirect* and a *direct* influence: an indirect influence through which the changed mass distribution influences the principal inertial axes of the Earth and thus also influences the position of the rotation axis in the Earth, and a direct influence in which the production of the mass transport consumes a part of the total impulse, and thus modifies in magnitude and direction the remaining impulse for the rotation of the Earth.

We first give a general depiction of the relevant relations.

The *indirect influence* of a mass transport is determined as follows.

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*) The corresponding air pressure differences are in no way large. If we imagine, for example, the total mass of 192,5 km^3 of mercury distributed uniformly on the spherical zone between 0° and 80° north latitude, there follows a covering of only 0,78 mm in height. The named mass excess on the northern spherical zone thus corresponds to a barometric state that is 0,78 mm higher in January than in July. In the same manner, the mass excess of 402,2 km^3 of mercury in the southern spherical zone corresponds to a mean barometric state that is 2,08 mm higher in July than in January.

**) Über Niveauschwankungen der Ozeane als eine mögliche Ursache der Veränderlichkeit der Polhöhe. Astron. Nachrichten 126 (1891), Nr. 3014.[253]

***) For more details on this subject, cf. Helmert, Die mathem. und physikalischen Theorien der höheren Geodäsie, II, Kap. 5, Leipzig 1884.[254]

The position of the moving mass with respect to the body of the Earth is regarded as known, and one calculates from this mass position the positions of the principal inertial axes of the Earth, and, in particular, the position of the inertia pole.[255] If the latter happens to coincide with the instantaneous rotation pole before the mass transport, it will differ during and after the transport. If one is allowed to assume that the Earth is a symmetric top with equal equatorial moments both after and before the transport, which is permissible to a very high degree of approximation, then the motion of the rotation pole after the mass transport consists of a rotation around the inertia pole. The period of this motion is—under the assumption of the previously calculated elasticity—fourteen months. The radius of the circle depends primarily on the velocity of the mass transport; the motion persists, theoretically, until it is changed by a new mass transport.

For the discussion of the *direct influence* of the mass transport on the impulse, we wish to assume that our mass transport is produced by internal forces, and therefore by forces that satisfy the law of equality of action and reaction inside the mass system of the Earth. Our fundamental impulse theorem of page 113 then holds just as unrestrictedly for the nonrigid Earth as for a rigid body (cf. a remark on page 111). This theorem states that the magnitude and direction of the total impulse of the mass system of the Earth remains constant in space. The total impulse is divided, however, into the impulse of the mass transport and that of the Earth's rotation. If the former be variable, then so must the latter be. In general, the rotation axis of the Earth and the position of the instantaneous pole of the Earth change with the impulse of the Earth's rotation. If the rotation axis coincides with the geometric pole before the mass transport, then it will be removed from it during the transport; if it originally moves in a circle about the geometric pole, then the radius of this circle is diminished or enlarged by the mass transport.

We give now some analytic developments, in which we first consider the two distinguished influences separately, and divide the subject matter into a series of individual problems.

First problem: a mass or the center of gravity of a not too extended mass system m is displaced from the position $X_0\,Y_0\,Z_0$ on the Earth to the position $X\,Y\,Z$. How does the polar principal inertial axis change?

Let the coordinate axes be the principal inertial axes for the original position of m. The principal moments of inertia are called A, $B = A$, C; the products of inertia (cf. page 98) are zero. For the changed position of m, we write the moments and products of inertia with respect to the coordinate axes in the form

$$\begin{aligned} \overline{A} &= A + a, & \overline{B} &= A + b, & \overline{C} &= C + c, \\ \overline{E} &= \quad\;\; e, & \overline{F} &= \quad\;\; f, & \overline{G} &= \quad\;\; g. \end{aligned}$$

The quantities $a, \ldots, e, \ldots$ have the meanings

$$\text{(1)} \qquad \begin{cases} a = m(Y^2 + Z^2 - Y_0^2 - Z_0^2), \ \ldots \\ e = m(YZ - Y_0 Z_0), \ \ldots, \end{cases}$$

and are treated as small quantities in comparison with A and C. For the determination of the changed position of the principal inertial axes, we begin, according to page 100, with the surface of the second order

$$(A + a)\xi^2 + (A + b)\eta^2 + (C + c)\zeta^2 - 2e\eta\zeta - 2f\zeta\xi - 2g\xi\eta = 1.$$

The principal axes of this surface, which are at the same time the principal inertial axes, are determined by the equations

$$\begin{aligned} (A + a - \lambda)\xi - g\eta - f\zeta &= 0, \\ -g\xi + (A + b - \lambda)\eta - e\zeta &= 0, \\ -f\xi - e\eta + (C + c - \lambda)\zeta &= 0. \end{aligned}$$

Here λ is to be chosen so that the three equations are compatible with one another. If this is done, then the ratios $\xi : \eta : \zeta$ determine the position of each of the three principal axes. It is convenient for the following to conceive ξ, η, ζ as the direction cosines of the relevant principal axis, and therefore to choose their absolute values so that $\xi^2 + \eta^2 + \zeta^2 = 1$.

We are particularly interested in the polar principal inertial axis, and may assume that this axis deviates only slightly from its original direction, the Z-axis. (For the equatorial principal inertial axes, the corresponding assumption would be impermissible, since their positions in the equatorial plane are originally undetermined, and thus can be changed significantly by a small mass transport.) We will therefore assume that ξ and η are small, and take ζ equal to 1. Quantities such as $f\xi$ and $a\xi$ are then to be struck; our third equation gives simply $\lambda = C + c$, and our first two equations become

$$\text{(2)} \qquad (A - C)\xi = f, \quad (A - C)\eta = e.$$

The change of direction of the principal axis in question is thus known on the basis of the values of f and e given in (1). The quantities ξ and

η can be regarded, at the same time, as the x- and y-coordinates of the inertia pole measured by the corresponding geocentric angles. If we multiply ξ and η by the radius R of the Earth, then we directly obtain the displacement of the inertia pole on the surface of the Earth.

In order to give a numerical example, we wish to assume that the mass m is displaced on a meridian, which we can take as the XZ-plane, from the latitude Θ_0 to the latitude Θ. Then

$$e = 0, \quad f = \frac{mR^2}{2}(\sin 2\Theta - \sin 2\Theta_0), \quad \eta = 0.$$

We can write the expression (2) for ξ as

$$\xi = \frac{A}{A - C}\,\frac{f}{A}.$$

The quantity A would be calculated for a homogeneous mass distribution as $2MR^2/5$, understanding by M the mass of the Earth; the actual mass distribution, however, corresponds better to the assumption $A = MR^2/3$.*) With use of the known numerical value of $A/(C - A)$, there follows

$$\xi = -456\,\frac{m}{M}\,(\sin 2\Theta - \sin 2\Theta_0).$$

If, for example, $\Theta_0 = -45°$ and $\Theta = +45°$, then the mass that is necessary to produce a deviation of the principal inertial axis by $1''$ is

$$m = \frac{\pi M}{180 \cdot 60 \cdot 60 \cdot 912} = \frac{1}{2}\,10^{-8}\,M.$$

The pole is naturally deflected in the same sense as the mass transport.

Second problem: The impulse of a mass displacement is given in magnitude and position with respect to the body of the Earth by a possibly time-dependent vector $\lambda\,\mu\,\nu$. It is assumed that the principal inertial axes are not changed by this mass displacement (see below). What influence does the mass displacement have on the position of the rotation axis?

If we disregard external forces and assume that the mass transport is caused only by internal forces, then the total impulse remains constant in space. This impulse has, with respect to the moving Earth frame, the components $L + \lambda$, $M + \mu$, $N + \nu$, where $L\,M\,N$ denotes the impulse of the Earth's rotation. According to page 140, the Euler equations obtain in the form

*) Cf. Helmert, l. c., II, p. 473.

$$\frac{d(L+\lambda)}{dt} = r(M+\mu) - q(N+\nu),$$
$$\frac{d(M+\mu)}{dt} = -r(L+\lambda) + p(N+\nu),$$
$$\frac{d(N+\nu)}{dt} = q(L+\lambda) - p(M+\mu).$$

We thus write

$$(3)\qquad \begin{cases} \dfrac{dL}{dt} = rM - qN + \mathsf{\Lambda}, \\ \dfrac{dM}{dt} = -rL + pN + \mathsf{M}, \\ \dfrac{dN}{dt} = qL - pM + \mathsf{N}, \end{cases}$$

in that we set

$$\mathsf{\Lambda} = -\frac{d\lambda}{dt} + r\mu - q\nu, \quad \mathsf{M} = -\frac{d\mu}{dt} - r\lambda + p\nu, \quad \mathsf{N} = -\frac{d\nu}{dt} + q\lambda - p\mu.$$

The quantities λ, μ, ν are small; we can thus replace p, q, r in the preceding equations by their approximate values in the unperturbed rotation of the Earth; that is, by the values $p = 0$, $q = 0$, $r = \omega$. The equations thus simplify to

$$(4)\qquad \mathsf{\Lambda} = -\frac{d\lambda}{dt} + \omega\mu, \quad \mathsf{M} = -\frac{d\mu}{dt} - \omega\lambda, \quad \mathsf{N} = -\frac{d\nu}{dt}.$$

The quantities $\mathsf{\Lambda}$, M, N are thus, just as the quantities λ, μ, ν are, known functions of time. Equations (3) may be interpreted as follows: our mass transport with impulse $\lambda\,\mu\,\nu$ influences the rotation of the Earth as if a given time-dependent turning-force $\mathsf{\Lambda}\,\mathsf{M}\,\mathsf{N}$ acted upon the body of the Earth.

Since we assume that the positions of the principal axes are not influenced by the mass transport, these axes are fixed in the body of the Earth, and we can set $L = Ap$, $M = Aq$, $N = Cr$ in (3); in addition, we can replace r by its approximate value ω in the terms that are accompanied by the small factor p or q. The first two of equations (3) are thus

$$(5)\qquad \begin{cases} A\,\dfrac{dp}{dt} = (A-C)\omega q + \mathsf{\Lambda}, \\ A\,\dfrac{dq}{dt} = (C-A)\omega p + \mathsf{M}. \end{cases}$$

The third equation is unimportant for the following.

Through multiplication by 1 and i, we combine equations (5) into a complex equation

$$(6)\qquad A\,\frac{d(p+iq)}{dt} = (C-A)i\omega(p+iq) + \mathsf{\Lambda} + i\mathsf{M},$$

and assume that the mass transport is periodic, so that λ, μ, ν, and therefore also $\mathsf{\Lambda}$, M, N, are periodic functions of time. We expand these functions in a Fourier series according to multiples of the period, and consider a single term of this series. We can thus assume for $\mathsf{\Lambda}+i\mathsf{M}$, in complete generality,

$$\mathsf{\Lambda}+i\mathsf{M} = ae^{i\alpha t} + a'e^{-i\alpha t}.$$

The corresponding general integral of (6) is then

$$p+iq = be^{i\alpha t} + b'e^{-i\alpha t} + ce^{i\beta t}, \tag{7}$$

where c is the constant of integration, and where we set, as abbreviations,

$$\begin{cases} \beta = \dfrac{C-A}{A}\omega, \\ b = \dfrac{a}{i\alpha A - i\omega(C-A)} = \dfrac{ia}{A}\dfrac{1}{\beta-\alpha}, \\ b' = \dfrac{a'}{-i\alpha A - i\omega(C-A)} = \dfrac{ia'}{A}\dfrac{1}{\beta+\alpha}. \end{cases} \tag{8}$$

The quantity c is in general complex, just as the previous a and a' are. The first two terms on the right-hand side of (7) represent the *forced* oscillation of the rotation axis caused by the mass transport, and the last term represents the *free* oscillation. The first terms naturally have the period of the mass transport, and the last term has the period of $A/(C-A)$ days indicated by the value of β. If we set the latter equal not to the Euler period, but rather to the Chandler period, then we consider in the simplest manner, in the sense of the previous section, the elastic compliability of the Earth, which is naturally relevant in this place as well.

As always in oscillation problems, we encounter here a certain resonance phenomenon; that is, a strengthening of the amplitude in the case of coincidence between the free and forced vibrations. This coincidence occurs, in our notation, if $\alpha = \pm\beta$, in which case either b or b' becomes infinitely large. We best measure the strengthening at a certain frequency by comparison with a very slow oscillation ($\alpha = 0$). According to (8), there follow, for the coefficient b_0 for a very small frequency and for the ratio of the coefficient b_α for an arbitrary frequency to that for a very small frequency,

$$b_0 = \frac{ia}{A}\frac{1}{\beta}$$

and

$$\frac{b_\alpha}{b_0} = \frac{1}{1-\alpha/\beta}. \tag{9}$$

(The same formula holds for the coefficient b' if we interchange $+\alpha$ with $-\alpha$; the following remarks that we attach to the value of b follow just as well from the corresponding formula for b' if we consider negative frequency, and therefore assign the opposite sense to the mass transfer.)

If, for example, the mass transport has the period of a year and we take the period of the free oscillation, as agreed, as the Chandler period, then $\alpha/\beta = 14/12$, and (disregarding the sign) $b_\alpha/b_0 = 6$. *The circumstance that the yearly period is not very far removed from the period of the pole oscillation has the consequence that a mass transport of a yearly period produces a sixfold stronger deviation than a process in which the same turning-force is applied infinitely slowly to the Earth.* If the mass transport, on the other hand, has a very short period (α very large), then α/β will be large and b_α/b_0 small. For example, we wish to refer to the significant mass transport that occurs relative to the rotating Earth due to the phenomenon of the tides.*) Here α/β is approximately equal to 840 and b_α/b_0 is approximately equal to $1/840$. *A mass transport of so short a period produces, for equal magnitude of the applied turning-force, only a vanishingly small effect on the rotation axis compared with an infinitely slow transport.* The mass system of the Earth is too inert to follow an influence with so short a period; it follows a perturbation all the more willingly and yieldingly as the disturbance period lies nearer to the natural period of the pole oscillation.

Moreover, the same resonance phenomenon occurs if we assume in the calculation, as in the first problem of this section, only the indirect action of the mass transport; that is, its influence on the mass distribution, in that the inertia pole of the Earth is also displaced periodically by a periodic mass transport, and thus an ever stronger oscillation of the rotation pole occurs as the period of the mass transport lies nearer to the natural period of the pole oscillation. We will have occasion to return to this in a third problem below.

Because of the small difference between the yearly period of the meteorological mass transports and the free oscillation period of the pole, the possibility exists, in any case, that a relatively weak meteorological

*) The mass transport considered here is indeed caused by external forces (the Moon attraction), so that the present discussion is not immediately valid.

mass transport can have a relatively strong pole oscillation as a result, a possibility that is to be kept in mind for the study of the pole oscillation of yearly period.

There is a class of mass transports for which the effect on the impulse that is treated here separately actually occurs separately, and the effect on the mass distribution vanishes. We speak of "a cyclic mass transport" if the displaced mass is immediately replaced by a new mass of the same density. Evidently, a cyclic mass transport gives no occasion for a displacement of the principal inertial axes, while it influences, on the other hand, the impulse of the Earth rotation according to the measure of its velocity and its quantity. This case permits of a very elegant treatment, particularly if the impulse of the mass transport remains constant with respect to the body of the Earth; it has been investigated in a series of works by V. Volterra.*)

It has not yet been possible, however, to demonstrate the existence of actual cyclic mass transports that are of sufficient intensity or sufficient duration to have a perceptible influence on the pole oscillations. In particular, Volterra's attempt to explain the pole oscillation with the Chandler period on this basis does not appear promising. The cyclical motions that Volterra must postulate in order to arrive at the Chandler period are of a purely hypothetical nature, and are not made probable by geophysical experience. Moreover, we will see in the following that the direct action of a mass transport on the impulse is generally small compared with its indirect action on the principal axes, and that a noncyclic mass transport generally influences the rotation of the Earth more than a cyclic mass transport of the same strength. Thus the Volterra investigations appear to have a more general mathematical interest than an immediate geophysical interest.

Purely theoretically, without consideration of geophysical questions, the motion of a top with an interior cyclic motion has been treated previously by A. Wangerin.**)

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*) Astronom. Nachr. Vol. 138 (1895), p. 33; Atti d. R. Accademia di Torino, Vol. 30 and 31 (1895). The explanations of G. Peano, ibid., are in the same direction. Volterra summarizes his investigations in Acta Mathematica, Vol. 22 (1898).[256]

**) Halle 1889, University publication. The problem is taken up in a mathematically generalized form by V. Volterra, Rend. d. R. Accademia dei Lincei (5) Vol. 4 (1895) and by E. Jahnke, Liouville's Journal (5) t. 5 (1899).[257]

Combining the developments given for our first and second problems, we now consider simultaneously the direct effect of a mass transport on the impulse and the indirect effect of the same mass transport on the mass distribution of the Earth. We pose, correspondingly, the following

Third problem: A mass m is displaced from an initial position $X_0\, Y_0\, Z_0$ in a specified manner, so that its coordinates X, Y, Z with respect to the body of the Earth are known, and are, in particular, periodic functions of time. The inertia pole of the Earth is thus deflected in a determined manner, and the impulse of the Earth's rotation will be influenced at the same time as if a determined turning-force $\mathsf{\Lambda\, M\, N}$ *acted on the body of the Earth. Construct and integrate the differential equations of the rotational motion.*

From the given time-dependent coordinates X, Y, Z of the mass m, we first compute the vector

$$mX', \;\; mY', \;\; mZ',$$

and next compute its moments

$$(10)\quad \lambda = m(YZ' - ZY'), \;\; \mu = m(ZX' - XZ'), \;\; \nu = m(XY' - YX')$$

about the coordinate axes, which are the components of the turning-impulse of the mass transport with respect to the same axes.

The motion of the body of the Earth will be represented after the transport as well as before, under the assumption that exterior forces are not present, by equations (3), in which $\mathsf{\Lambda}$, M, N can be calculated with sufficient accuracy by means of equations (4) with the just-given λ, μ, ν. In fact, equations (3) of page 712 or equations (2′) of page 140, from which we deduced equations (3), are valid for an arbitrary rectilinear system of axes that are fixed in the top, whether or not these axes are the principal inertial axes. In contrast to the considerations for our second problem, our coordinate axes are now no longer the principal inertial axes; if we assume, for example, that they were the principal axes at the beginning of the motion, they will lose this property to the extent that the inertia pole is deflected by the mass transport. As a result, there appear in the place of the simple relations $L = Ap$, $M = Aq$, $N = Cr$ the general equations (2) of page 95 for the relation between the impulse vector and the rotation vector, which we can write, with consideration of the definitions of the quantities $a\, b\, c$, $e\, f\, g$ in equations (1), as

$$\begin{aligned} L &= (A+a)p - gq - fr, \\ M &= -gp + (A+b)q - er, \\ N &= -fp - eq + (C+c)r. \end{aligned}$$

We consider $a\,b\,c$, $e\,f\,g$, $p\,q$ as small quantities, and can thus simplify the previous equations as

$$L = Ap - fr, \quad M = Aq - er, \quad N = (C + c)r. \tag{11}$$

We must insert these values into equations (3). From the third of these equations, it first follows that dr/dt (in contrast to r itself) will be a small quantity, which one could also deduce from the fact that the unperturbed original motion consists of a *uniform* rotation $r = \omega =$ const. In the first two equations of (3), we further neglect all those terms of the second order in the small quantities, and replace r by its approximate value ω in the terms of the first order. There follow

$$\begin{cases} A\dfrac{dp}{dt} = (A - C)q\omega + \mathsf{\Lambda}', \\ A\dfrac{dq}{dt} = (C - A)p\omega + \mathsf{M}', \end{cases} \tag{12}$$

where the abbreviations

$$\begin{cases} \mathsf{\Lambda}' = \omega\dfrac{df}{dt} - \omega^2 e + \mathsf{\Lambda}, \\ \mathsf{M}' = \omega\dfrac{de}{dt} + \omega^2 f + \mathsf{M} \end{cases} \tag{13}$$

are used.

Equations (12) have completely the same form as equations (5); the quantities $\mathsf{\Lambda}'$, M' here, just like $\mathsf{\Lambda}$, M there, are known functions of time if the time dependence of the mass transport is known. The quantities $\mathsf{\Lambda}'$, M' bring together the direct action on the impulse and the indirect action of the mass transport, and may be interpreted once more as an apparent turning-force acting on the body of the Earth. The analytic expression

$$\begin{aligned} \mathsf{\Lambda}' &= -\frac{d}{dt}(\lambda - \omega f) + \omega(\mu - \omega e), \\ \mathsf{M}' &= -\frac{d}{dt}(\mu - \omega e) - \omega(\lambda - \omega f) \end{aligned} \tag{14}$$

of this apparent turning-force, which results immediately from the defining equations (4) and (13) for $\mathsf{\Lambda}$, M and $\mathsf{\Lambda}'$, M', is also noteworthy. In order to consider the indirect effect of the mass transport in addition to the direct effect, one must simply replace λ, μ by $\lambda - \omega f$, $\mu - \omega e$.

The further treatment of equations (12), their integration and the discussion of their solution, is no different from the above treatment of equations (5); in particular, the above-emphasized resonance phenomenon occurs if the mass transport is periodic and its period lies

near the period of the free oscillation of the Earth's axis.

We first wish to decide the question whether, for a periodic mass transport, the direct or the indirect—that is, the effect on the impulse or the effect on the mass distribution—is more significant, in order to arrive at a further simplification of equations (12) that is useful for numerical calculation. We have, for this purpose, merely to examine, according to equations (14), the ratio of the pairs λ, μ and ωf, ωe.

The rule according to which the mass transport runs temporally in the body of the Earth may be expressed by the most conveniently chosen equations

$$\begin{aligned} X &= X_0 \ + a \sin \alpha t, \\ Y &= Y_0 \ + b \sin \alpha t, \\ Z &= Z_0. \end{aligned}$$

The mass in question thus oscillates about its initial and mean position $X_0\,Y_0\,Z_0$ with the period $2\pi/\alpha$. We calculate, according to (1),

$$e = mZ_0 b \sin \alpha t, \qquad f = mZ_0 a \sin \alpha t,$$

and, according to (10),

$$\lambda = -mZ_0\, b\alpha \cos \alpha t, \qquad \mu = mZ_0\, a\alpha \cos \alpha t.$$

There follow the ratios

$$\frac{\lambda}{\omega f} = -\frac{\alpha}{\omega}\,\frac{b}{a}\,\frac{\cos \alpha t}{\sin \alpha t}, \qquad \frac{\mu}{\omega e} = \frac{\alpha}{\omega}\,\frac{a}{b}\,\frac{\cos \alpha t}{\sin \alpha t}.$$

We do not wish to make a detailed assumption about the amplitudes a and b; we will assume, however, that they are of the same order of magnitude. The orders of magnitude of the preceding ratios are then given in the mean by the factor α/ω. Now the quantities α and ω are inversely proportional to the period of the mass transport and the period of the rotation of the Earth, respectively. The ratio α/ω will thus be equal to the reciprocal number of days of the period of the mass transport. We saw previously that only mass transports with a period that lies near the natural period of the pole oscillation can exert a strong influence on the pole oscillation. Thus for all mass transports that interest us, α/ω is a small number; for the meteorological mass transports, for example, it is equal to 1/365. It follows that *for these mass transports the direct action is considerably smaller than the indirect action*, so that we can strike λ and μ in equations (14) with respect to ωe and ωf. At the same time, we can also strike df/dt with respect to ωe and de/dt with respect to ωf, since the ratio of these pairs of quantities is

again determined by the value α/ω. Equations (4) simplify, on the basis of these omissions, to

$$\mathsf{\Lambda}' = -\omega^2 e, \quad \mathsf{M}' = +\omega^2 f,$$

where we can also write, according to equations (2), if we introduce the angular deviations of the principal axes,

$$\mathsf{\Lambda}' = -\omega^2(A-C)\eta, \quad \mathsf{M}' = +\omega^2(A-C)\xi. \tag{15}$$

This simplification is derived here on the basis of a very special assumption about the mass transport. One easily sees, however, that analogous conclusions are also possible for a more general assumption in which one gives each of X, Y, Z by a Fourier series, to whose first terms we have restricted ourselves above; for the terms with a longer period (that is, long compared to the period of the rotation of the Earth), the indirect influence caused by the change of the mass distribution dominates, while for the terms of shorter period (that is, short compared to the free oscillation of the Earth's axis), the direct influence on the impulse as well as the indirect influence on the Earth's rotation will be insignificant. The direct influence may be decisive only for sudden mass transports, and it indeed remains doubtful whether such mass transports with significant strength are present in reality.

If we insert the values (15) into our differential equations (12), then these equations become

$$\begin{cases} \dfrac{dp}{dt} = \dfrac{A-C}{A}\,\omega(q-\omega\eta), \\ \dfrac{dq}{dt} = \dfrac{C-A}{A}\,\omega(p-\omega\xi). \end{cases} \tag{16}$$

Here we wish to introduce, in addition to the coordinates ξ, η of the inertia pole, the similarly measured coordinates of the rotation pole, which may be called u, v. The coordinates u, v signify the direction cosines of the rotation axis with respect to the coordinate axes X and Y, and are therefore equal to p/r and q/r, respectively, where we can also take, with sufficient accuracy, p/ω and q/ω. If we also use, as in equation (8), the abbreviation β for the frequency of the free oscillation of the axis of the Earth, then our equations become

$$\begin{cases} \dfrac{du}{dt} = -\,\beta(v-\eta), \\ \dfrac{dv}{dt} = +\,\beta(u-\xi). \end{cases} \tag{17}$$

These equations simply state *that the rotation pole will rotate at each instant about the inertia pole with angular velocity* β. The sense of the

rotation coincides with the sense of the rotation of the Earth; the coordinate system is imagined to be chosen so that the positive X-axis is transformed into the positive Y-axis along the shortest path by the rotation of the Earth.

For the purpose of integration, we combine equations (7) in complex form as

$$\frac{d(u+iv)}{dt} = i\beta\big((u+iv)-(\xi+i\eta)\big), \tag{18}$$

and assume that the inertia pole executes an elliptical oscillation about its mean position as a result of the mass transport. We can then make for $\xi + i\eta$, as previously for $\mathsf{\Lambda} + i\mathsf{M}$, the assumption

$$\xi + i\eta = ae^{i\alpha t} + a'e^{-i\alpha t}. \tag{19}$$

There corresponds as the particular integral of (18), which represents the oscillation that is *forced* by the mass transport (we can disregard in the following the *f r e e* oscillation with the period $2\pi/\beta = 14$ months),

$$\begin{cases} u + iv = be^{i\alpha t} + b'e^{-i\alpha t}, \\ b = \dfrac{a\beta}{\beta-\alpha}, \quad b' = \dfrac{a'\beta}{\beta+\alpha}. \end{cases} \tag{20}$$

Equations (20) represent, just as equation (19) does, an elliptical oscillation. In order to conveniently visualize the relative position and magnitude of the two ellipses, we can imagine the coordinate directions chosen so that they coincide with the principal axes of the ellipse (19). Then a and a' are real, and, according to (20), b and b' are also real. If we name the principal axes of the two ellipses h, k and H, K, respectively, then we can write, instead of (19) and (20),

$$\xi + i\eta = h\cos\alpha t + ik\sin\alpha t, \quad h = a + a', \quad k = a - a', \tag{19'}$$

$$u + iv = H\cos\alpha t + iK\sin\alpha t, \quad H = b + b', \quad K = b - b'. \tag{20'}$$

One thus recognizes that the directions of the principal axes of the two ellipses coincide; for what concerns their magnitude, it follows from (20) and the definitions of h, k, H, K that

$$H = \frac{\beta(\beta h + \alpha k)}{\beta^2 - \alpha^2}, \quad K = \frac{\beta(\beta k + \alpha h)}{\beta^2 - \alpha^2}. \tag{21}$$

It is noted that H, K are to be calculated with signs, and that one must also, if necessary, bestow on h a negative sign in order to express by (19′) the correct rotation sense of the inertia pole. The inversion of equations (21) gives

$$h = H - \frac{\alpha}{\beta}K, \quad k = K - \frac{\alpha}{\beta}H. \tag{22}$$

We may first give some numerical examples and figures. We assume that the relevant mass transport is of meteorological origin, and thus has the period of one year. In order to include the elasticity of the Earth into the calculation (cf. page 713), we regard the period of the free oscillation as the Chandler period. Then α/β is approximately equal to 7/6. The inertia pole may execute a linear oscillation, and therefore, for example, $h = 0$ and $\eta = k \sin \alpha t$. From (21) there follows

$$H = -\frac{42}{13}\,k = -3{,}2\,k; \quad K = -\frac{36}{13}\,k = -2{,}8\,k,$$

and from (20′),

$$u = -3{,}2\,k\,\cos \alpha t, \quad v = -2{,}8\,k\,\sin \alpha t.$$

This case is illustrated in Fig. 107a. We have designated the corresponding points of the inertia pole and the rotation pole (that is, points occupied at the same time) with equal numbers.

The previous assumption with respect to the trajectory of the inertia pole is retained in Fig. 107b. We have chosen, however, the Euler period

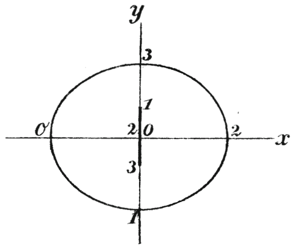

Fig. 107 a.

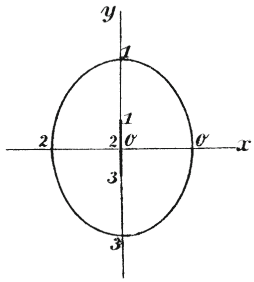

Fig. 107 b.

as the period of the free oscillation, as would be the case for an absolutely rigid Earth. Then $\alpha/\beta = 5/6$, and

$$H = \frac{30}{11}\,k = 2{,}7\,k; \quad K = \frac{36}{11}\,k = 3{,}3\,k,$$

$$u = +2{,}7\,k\,\cos \alpha t, \quad v = +3{,}3\,k\,\sin \alpha t.$$

Both Fig. 107a and Fig. 107b express the repeatedly emphasized resonance effect, due to which the motion of the rotation pole will be essentially more extensive than that of the inertia pole when the periods of the free and forced oscillations are not very different. That the rotation pole is found on the side opposite the inertia pole in Fig. 107a and on the same side in Fig. 107b (has opposite or equal phase) corresponds to a general rule of oscillations: the opposite phase always occurs in the case $\alpha > \beta$, and the same phase in the case $\alpha < \beta$. The passage between the ellipses is mediated by the case $\alpha = \beta$, where

our ellipse (see equations (21)) is transformed into a circle of infinitely large radius. In the case $\alpha = 0$ (infinitely long period, or secular mass transport) the elliptical oscillation degenerates into a linear oscillation, since (cf. (21)) $H = 0$, $K = k$; the rotation pole then follows exactly the path of the inertia pole. In the case $\alpha = \infty$ (infinitely rapid oscillation), the rotation pole may not at all follow the action of the mass transport; here, according to (21), $H = K = 0$. If one imagines in Fig. 107b that the degenerate line of the inertia pole is transformed into the infinite circle through a continuous series of broadened ellipses, to which the ellipse of the rotation pole constructed in this figure also belongs, and in Fig. 107a that this infinite circle is transformed through a continuous series of narrowed ellipses, of which one coincides with the ellipse drawn in this figure, then one has the complete image of the possible trajectories of the rotation pole for arbitrary values of the ratio α/β.

The relations are no longer so transparent, however, if we assume that the trajectory of the inertia pole itself is elliptic, and therefore add to the just considered linear oscillation a second oscillation perpendicular to it and shifted in phase. It can then occur, in particular, that the resonance will be concealed, to a certain extent, by interference; the multiplicity of the relative positions of the two ellipses that are possible according to equations (21) will then become extraordinarily large. —

After dispatching the three posed problems, we now return to the actual conditions that are present for the Earth, and, in particular, to the air mass transport that was discussed at the beginning of this section. As Mr. S p i t a l e r calculates on the basis of the air pressure maps (by mechanical quadrature over the surface of the Earth), air transport will deflect the inertia pole

in January by $0'',055$ toward $100°$ west of Greenwich,

in July „ $0'',041$ „ $68°$ east of Greenwich.

The inertia pole thus moves at these two times by approximately equal angles toward approximately opposite meridians. The amplitudes for April and October are not calculated, but rather only estimated; they are directed approximately toward the meridians 180° and 0° and are presumably smaller than those given above. The inertia pole therefore runs in the direction from east to west; that is, in the opposite sense from the rotation of the Earth. The exact form of the trajectory may not

be established from these data, and it is thus not possible to determine the corresponding path of the rotation pole.

The reverse path, in contrast, is passable. According to page 682, the rotation pole oscillation of yearly period is described as an ellipse with principal axes $0'',104$ and $0'',044$, whose major axis is directed toward the meridian 19° east of Greenwich, and which is traversed in the sense of the rotation of the Earth. We thus set $H = 0'',104$, $K = 0'',044$, and calculate, according to equations (22) with $\alpha/\beta = 7/6$,

$$h = 0'',053, \quad k = -0'',077.$$

The so-determined ellipse will (due to the sign of k) be traversed in the sense opposite to that of the previous ellipse; the positions of the major and minor axes are opposite to those of the previous ellipse.

The ellipse H, K of the rotation pole and the theoretically corresponding ellipse h, k of the inertia pole are drawn in Fig. 108. Corresponding positions of the two ellipses are marked by the same month names. Further, the position of the inertia pole and its sense of motion according to the calculations of Spitaler for the times January and July are registered in the figure. The relevant points are made recognizable by small circles. One sees from the figure that a general agreement is present between these points and the theoretically determined simultaneous positions of the inertia pole, at least in order of magnitude. The actual differences in their positions can be explained either by our still rather complete ignorance of the arctic air pressure values, or by the fact that other meteorological processes (water transport, etc.) in addition to air transport influence the yearly trajectory of the rotation pole.

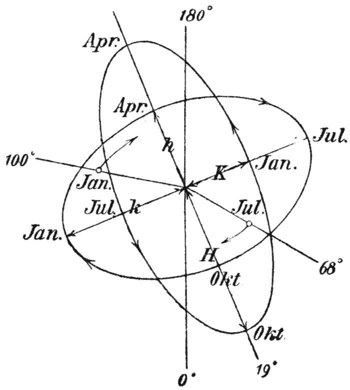

Fig. 108.

All in all, there is a good basis for the assumption that it will be possible, with further enrichment of the observational data, to satisfactorily explain the yearly component of the pole oscillations from meteorological mass transports.

Not as favorable stand the prospects for the explanation of the remainder of the aperiodic pole oscillations that are represented in Fig. 106. Secular mass transports of reasonably probable amounts give, at

most, only very small effects on the inertia and rotation poles.*) Also, the general form of Fig. 106, in so far as we can speak of it with real significance, gives the impression that it consists of aperiodic pole oscillations of longer or shorter duration, followed by perturbations acting in the reverse sense.

Perturbations of this character would be produced, in our previous terminology, by the direct influence on the impulse of the rotational motion if a mass transport on the Earth occurred rather suddenly and then came again to rest, so that the impulse of the mass displacement is first generated and then annihilated, and the corresponding impulse change of the Earth's rotation occurs first in one and then in the opposite sense. Since we generally have, however, no indication for the assumption that such mass transports of sufficient strength are possible on the Earth, we consider it useless to carry out the just indicated representation further. —

With respect to the general analytic developments of this section, it is emphasized that for the treatment of the body of the Earth with a variable mass distribution, the fundamental equations (3) result immediately from our conception of the Euler equations, even for the cases in which the coordinate axes are not or do not remain the principal axes of the body of the Earth. With the establishment of the chosen coordinates, we then arrived at the simplest equations (17) by mere specialization to the particular present case. The problem is treated most thoroughly in the literature by G. H. Darwin.**) Darwin does not, as we do, adopt coordinate axes that are fixed in the body of the Earth, but rather the moving instantaneous principal axes in the body of the Earth, and likewise arrives in this manner at the final equations (17). The calculations at the basis of Fig. 107b were first given by R. Radau,***) on which account the ellipse of that figure is occasionally designated as the Radau ellipse. F. R. Helmert†) has discussed the relation between the ellipses of the inertia pole and the rotation pole under more general assumptions.

Our presentation of the pole oscillations would be incomplete, however, if we did not mention, in addition to the *centrifugal* effect of the

*) Cf. Tisserand, Mécanique céleste II, Chap. 29, art. 208 and Chap. 30, art. 218.

**) G. H. Darwin: On the influence of Geological Changes on the Earth's Rotation. London, Phil. Trans. 167 (1877), with an appendix by Lord Kelvin.

***) R. Radau, Comptes Rendus 111 (1890) and Bulletin Astronomique 7 (1890).

†) F. R. Helmert, Astronom. Nachr. 126 (1891), Nr. 3014.

mass transports on the rotation pole, certain *centripetal* tendencies that are caused by the appearance of frictional influences, and which, in a certain manner, can calm and simplify the motion of the rotation pole, just as the previous influences disturb and complicate it.

We think, in the first place, of the friction associated with the *tides*, and, in particular, the usual tides produced by the Moon or Sun attraction. Immanuel Kant emphasized the presence of such friction as early as 1754, and thus deduced the necessity of a secular elongation of the sidereal day.[258] We need not discuss in detail how this friction occurs;[*]) the following somewhat grotesque representation suffices for our purpose. Two diametral tidal bulges accumulate in the water that covers the surface of the Earth; the Earth continues to rotate under them, while the tidal bulges themselves stand still, or, according to the measure of the motion of the Moon, slowly change their relative position. Through the viscosity of the water, the bulges apply a turning-moment on the Earth that opposes its rotation. If the moon stood exactly fixed in the instantaneous equator of the Earth and the symmetry of the tidal motion were not disturbed by the continents, the axis of the turning-moment would coincide with the instantaneous rotation axis, and its magnitude would be proportional to the magnitude of the rotation. We wish to regard this simplest imaginable determination of the turning-moment of the tidal friction as approximately and in the mean valid. We can, for example, compare the two tidal bulges with the two shoes of a railroad brake that are applied to the rotating wheel and slow its rotation.

The further consequence of the influence of tidal friction is thus reduced to a top problem that was already treated in Chap. VII, §7 as the problem of air resistance: a force-free top stands under the influence of a turning-force whose axis is the instantaneous rotation axis, and whose magnitude is negatively proportional to the instantaneous rotation. We saw that the rotation of such a top is gradually annihilated, and that, at the same time, the rotation axis spirals asymptotically to the axis of the greatest principal moment of inertia (cf. page 588 and the figure of page 589). For the Earth, the axis of the greatest moment of inertia is

[*]) Cf. Chaps. 16 and 17 of the work of G. H. Darwin for further literature. In particular, the astonishing cosmological effects of tidal friction are indicated there.[259]

the polar principal axis. It may thus appear that tidal friction is a countervailing and damping effect on the pole oscillations, and that we owe it to this effect that, in spite of temporary perturbations, the rotation pole remains, empirically, so near in the mean to the inertia pole.

However, a numerical calculation shows that this effect is entirely to be neglected. We begin from equations (1) and (6) of pages 587 and 588. In equation (6), β signifies the angle that is enclosed by the instantaneous rotation axis and the axis of the greatest principal moment of inertia at time t, and β_0 is the same angle at $t = 0$. If we restrict ourselves to small angles β and β_0, we can write equation (6) as

$$\frac{\beta}{\beta_0} = e^{-\lambda t\left(\frac{1}{A} - \frac{1}{C}\right)} = e^{-\frac{\lambda t}{C}\varepsilon},$$

where ε (approximately equal to 1/300) denotes, as previously, the ellipticity of the Earth. According to the cited equation (1), on the other hand,

$$\frac{r}{r_0} = e^{-\frac{\lambda t}{C}}.$$

The two equations combined give

$$\frac{\beta}{\beta_0} = \left(\frac{r}{r_0}\right)^{\varepsilon}.$$

Thus in the time that the tidal friction reduces an originally present deviation β_0 of the rotation axis to half its amount $\left(\beta = \frac{1}{2}\,\beta_0\right)$, it reduces, at the same time, the originally present rotation of the Earth r_0 by the fraction

$$\left(\frac{1}{2}\right)^{1/\varepsilon} = 2^{-300} = \frac{1}{2}\cdot 10^{-90}$$

of itself. In other words, *the rotation of the Earth must, due to tidal friction, have come as well as completely to rest before half of the originally present deviation of the rotation axis is canceled.* In such a manner, it is understood that tidal friction does not at all come into consideration for the question of the pole oscillations (equally little as the mass transports of the usual Moon and Sun tides; cf. page 714), and also cannot (cf. page 593) be drawn upon for the explanation of the secular changes of the rotation axis, as has frequently been postulated in geology.

There is, however, yet another type of flow and another type of flow friction that may deflect the rotation pole more effectively back toward the inertia pole; namely, the flow that is produced by the pole oscillation itself (cf. page 684, where we discussed, in particular, the

fourteen-month component of this flow). This flow also will be accompanied by friction, and indeed one can assume here that the friction acts against the *change of the rotation axis*, and that its axis stands *perpendicular* to the rotation axis, while the friction for the usual Moon and Sun tides depends on the instantaneous magnitude of the *rotation itself*, and its axis *coincides* with the instantaneous rotation axis.

If we wish to form the simplest possible, if again somewhat rough, representation for the formation of this flow, then we can speak in the following manner. The position of the rotation axis in the body of the Earth is specified at a given time by the quantities p, q, r; this position corresponds, if the effect of the continents is disregarded, to a disposition of the water covering in which a fluid belt is formed about the instantaneous equator that is perpendicular to the rotation axis. At a following point of time, the position of the rotation axis will be given by $p + p'\,dt$, $q + q'\,dt$, $r + r'\,dt$; the fluid belt now lies about the new equator, and is rotated with respect to its previous location. We transform the belt from its first to its second position if we rotate it about the common perpendicular to the first and second positions of the rotation axis, and indeed through an angle that is equal to the deviation angle of the rotation axis. The fluid friction opposes this rotation; we assume for simplicity that the moment of the fluid friction acts about the same axis and is proportional to the magnitude of the rotational velocity. The axis of the fluid friction is then calculated by the subdeterminants of the schema

$$\begin{vmatrix} p & q & r \\ p' & q' & r' \end{vmatrix}.$$

The components of the fluid friction will thus be proportional to the expressions

$$qr' - rq', \quad rp' - pr', \quad pq' - qp'.$$

If we consider that the quantities p, q, p', q', r' are small, and that r is approximately equal to ω, then we can write, with the omission of small quantities of the second order,

$$-\omega q', \quad \omega p', \quad 0.$$

With the use of a positive factor of proportionality λ, we correspondingly set the components of the fluid friction equal to the quantities

$$-\lambda A q', \quad +\lambda A p', \quad 0.$$

In fact, one easily recognizes that through this assumption, the above agreements on the magnitude, axis, and sense of the frictional moment will be answered if the rotational velocity of the Earth ω is calculated as positive, and the coordinate axes therefore have the position given on page 720. The moment of inertia A was added to the previous expressions as a factor so that the quantity λ has the dimension of a pure number, as is convenient for the sequel.

In order to determine the influence of this fluid friction on the pole oscillations, we return to the Euler equations, to which we add, on the right-hand side, the just-determined components of the fluid friction. In the first approximation, the equation for the component r will not be changed. We can thus regard this component as a constant even with the consideration of friction, and set it equal to ω; in other words, the length of the sidereal day will not be increased by the friction now in question, within the bounds of precision established by us. The Euler equations for the components p and q of the rotation vector are, if we disregard the perturbation of the motion by the mass transport and consider only the free oscillation of the Earth's axis,

$$Ap' = (A - C)\omega q - \lambda A q',$$
$$Aq' = (C - A)\omega p + \lambda A p'.$$

For the purpose of integration, we combine these equations, in the frequently written manner, into the complex equation

$$A(p' + iq') = (C - A)i\omega(p + iq) + i\lambda A(p' + iq'),$$

where we can also write, with the introduction of the ellipticity ε,

$$(1 - i\lambda)(p' + iq') = \varepsilon i\omega(p + iq).$$

The number λ will, in any case, be small compared with 1, since in the opposite case a periodic pole oscillation could generally not exist. Thus we can also recast the equation without perceptible error as

$$\frac{p' + iq'}{p + iq} = \varepsilon i\omega(1 + i\lambda),$$

and integrate it as

$$p + iq = a\,e^{-\varepsilon\omega\lambda t + \varepsilon i\omega t}.$$

The quantity a is the constant of integration, which depends on the initial position of the Earth's axis; that is, on the consideration of previous perturbations.

It now follows that the friction leaves the period of the pole oscillation unchanged (unchanged up to quantities of the second order); its

frequency will here too be determined by the product $\varepsilon\omega$. In contrast, the oscillation is now, because of the friction, *damped*. The damping factor for the duration of a free oscillation is, according to the preceding formula, equal to $e^{-2\pi\lambda}$. Because of this damping, the rotation pole will evidently approach the inertia pole; it is also clear that the previously emphasized resonance effect will be mitigated, so that the amplitude of the rotation pole will no longer become infinitely large for coincidence of the free and forced oscillations, but rather will take on a finite value determined by the value of the damping factor.

We are unfortunately in complete uncertainty concerning the numerical magnitude of the damping, and in particular concerning the damping constant λ. Since we were previously able to say nothing about the magnitude of the relevant tides (cf. page 706), it will be even less possible to numerically estimate the magnitude of their frictional effect.

We wish to remark that the deformation of the Earth discussed in the previous section is very probably accompanied by an energy loss, and will thus likewise provide a contribution to the damping of the free oscillation. At least there is no elastic body known to us in which an initially generated deformational oscillation does not soon expire; we attribute this circumstance to the occurrence of interior frictional processes or elastic aftereffects. It would be most highly unphysical to assume that it should be otherwise for the body of the Earth. As a result, it appears fitting to consider interior friction of the Earth, in addition to tidal friction, as a possible cause of damping for the pole oscillations.

Until the present time, the damping effect of the different possible energy losses (which is indeed properly considered in analogous cases for other mechanical problems*)) has always been neglected in the calculational treatment of the pole oscillations, in that the pole trajectory is represented by a Fourier series that progresses in pure, undamped trigonometric functions of time (cf. the citations to *Chandler* on page 673 and *van de Sande Bakhuyzen* on page 682). Our graphical reduction of the pole trajectory in §6 of this chapter is also based on this assumption, and is to be modified if we would consider damping, or if we wished to identify the magnitude of the damping as well as the different periods

*) Cf., for example, R o u t h, Dynamik starrer Körper, Bd. II (German edition, Leipzig 1898) Kap. VII §331–333.

hidden in the experimentally observed pole trajectory. Since a predictive theoretical calculation of the damping constant λ seems rather hopeless, one should perhaps seek, in the manner just indicated, to obtain a result from the pole oscillation itself.

The preceding representation of the effect of the pole oscillation tides is naturally quite idealized; because of the influence of the continents on the tidal motion, the phenomena will be much more complicated in reality. It is thus desirable to determine the effect of an arbitrary energy-dissipating circumstance, or at least its sense, by an entirely general consideration, without particular assumptions, that also includes the case of deformation friction in the interior of the Earth.

For the pole oscillation and the flows and deformations generated by it, as well as for the corresponding flow friction and deformation friction, only internal forces come into play. These forces leave the total impulse of the mass system, which we call the Earth, unchanged (in contrast to the previously considered flow friction that is produced by the external attractive forces of the Sun and Moon). For the components of the total impulse, the equation

$$L^2 + M^2 + N^2 = \text{const.}$$

thus obtains, which we can interpret as the equation of a sphere. On the other hand, the *vis viva* of the system will be diminished by friction, in that a portion of the *vis viva* will be transformed into heat. If we permit ourselves to carry over the expression for the *vis viva* of the rigid top to our moving system, then we can write

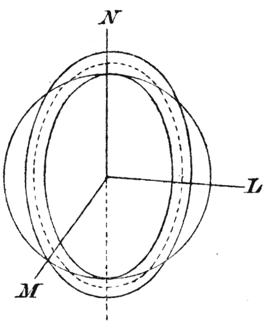

Fig. 109.

$$\frac{L^2 + M^2}{A} + \frac{N^2}{C} = 2T,$$

and can interpret this equation, for any value of T, as an ellipsoid of revolution in the coordinates L, M, and N. And indeed this equation represents a prolate ellipsoid of revolution (because $C > A$), which, remaining similar to itself, gradually contracts (because of the gradual decrease of T). The endpoint of the impulse vector L, M, N must lie on the intersection curve of the two surfaces (sphere and ellipsoid); this intersection curve contracts, however, to a point on the N-axis as our ellip-

soid is gradually diminished (cf. Fig. 109). The impulse vector and the rotation axis thus simultaneously approach the polar principal axis of inertia, and the state of motion goes over into a simple uniform rotation about this axis.

In so far as this deliberation is applicable to the case of tidal friction or other dissipative influences, we may claim that such influences counteract any induced disturbance of the simplest state of motion of the Earth, and that the position of the rotation pole will be stabilized on the surface of the Earth.

§9. The proof of the rotation of the Earth by the top-effect. Foucault's gyroscope and Gilbert's barogyroscope.

Having carried out in the year 1851 his brilliant pendulum experiment for the proof of the rotation of the Earth, Léon Foucault undertook in the following year to accomplish the same purpose with the top-effect. He used a top in a Cardanic suspension (cf. the schematic Fig. 2 of page 2), whose individual components (rotor, inner and outer rings) were adjusted with great care, so that the intersection of their axes of rotation was simultaneously the center of gravity of each component. Foucault's experimental disposition was twofold: in the first case,*) he gave the top *three degrees of freedom*, in that he let the outer ring rotate on conical bearings about a vertical axis. These bearings served not so much for supporting the top as for the prevention of its lateral motion; the weight of the top was borne by a torsionless thread from which the outer ring was suspended. The inner ring was supported by knife edges that rested on bearing surfaces fixed to the outer ring.[260] In the second case,**) he held the inner ring fixed with respect to the outer, and thus operated with a top of only *two degrees of freedom*, which, due to its attachment to the Earth, was constrained to move in a certain manner.

According to Foucault, the rotor of the top with three degrees of freedom and a strong rotation retains *the original direction of its axis in absolute space*; or, otherwise expressed, this axis is constantly directed *to the same point of the firmament.* As seen from the Earth, therefore,

*) Sur une nouvelle démonstration expérimentale du mouvement de la Terre, Comptes Rendus t. 35, Paris 1852, p. 421.

**) Sur le phénomènes d'orientation des corps tournants entraînés par une axe fixe à la surface de la Terre—Nouveau signes sensibles du mouvement diurne. l. c. page 424.[261]

each of its points moves parallel to the direction of the equator. Geometric considerations of the simplest kind then show the correctness of the following statements.

If the axis points toward the zenith at the beginning of the experiment, then it forms the angle $\omega \cos\varphi\, \Delta t$ (ω = the angular velocity of the rotation of the Earth, φ = the geographic latitude) with the plumb line after the observation interval Δt, since the original zenith describes this arc about the pole of the heavens in the same time interval. If, on the other hand, the axis of the rotor is originally horizontal and in the direction of the meridian, then it remains horizontal for a sufficiently short observation time, and forms the angle $\omega \sin\varphi\, \Delta t$ with the meridian after the time Δt, since a star on the horizon in the direction of the meridian has the polar angle φ (or $\pi - \varphi$) and describes an arc $\omega \sin\varphi\, \Delta t$ in the horizontal direction during the time Δt. The same expression $\omega \sin\varphi\, \Delta t$, which, moreover, also appears for the Foucault pendulum, is valid for the horizontal component of the angular change for an arbitrary horizontal initial position of the rotor axis. If we ask, namely, for the apparent motion of a star on the horizon at an arbitrary azimuth, then this motion consists of a rotation ω about the polar axis, which we can decompose into a rotation $\omega \sin\varphi$ about the plumb line and a rotation $\omega \cos\varphi$ about the meridian. The former component produces the horizontal motion of the star, which will thus amount to $\omega \sin\varphi\, \Delta t$ during the observation time Δt; the latter component gives the elevation change of the star. The former component, and thus also the horizontal motion of the axis of the top, is independent of the azimuth of the initial position.

The experimental disposition of Foucault is tied to the latter circumstance. One notices that for a horizontal initial position, the motion of the axis of the top is decomposed by the Cardanic suspension itself into its two components; the motion of the outer ring reproduces the horizontal component of the motion of the axis of the top, while the motion of the inner ring determines the change in the elevation of the axis of the top.[262] Foucault used a microscope to observe the outer ring, whose rotation should be equal to $\omega \sin\varphi\, \Delta t$. Foucault gives 8 to 10 minutes as the largest possible value of the observation time. If we therefore calculate the expected deflection with $\Delta t = 8$ min. and $\varphi = 49^\circ$ (the approximate latitude of Paris), there follows, in degree measure,

$$\omega \sin\varphi\, \Delta t = \frac{360 \cdot 8}{24 \cdot 60}\, 0{,}75 = 1^\circ{,}5.$$

It must surely be possible to detect this rather considerable rotation under the microscope.

This contrasts, to a certain extent, with the circumstance that Foucault speaks only of the direction of the rotation, which resulted correctly in his experiment, and therefore gave the direction of the rotation of the Earth in the opposite sense, but does not give numerical values for his observations. We know not to what extent these observations coincide with the theoretical values. As long, however, as a quantitative agreement is not proven, or as long as the sources of error that cause the disagreement are unknown, the experiment can hardly be claimed as an irrefutable proof of the rotation of the Earth; it could indeed be, in the present case, that the sources of error influence the deflection more strongly than the rotation of the Earth itself, and that the correct sense of the result is produced only speciously by an accidental grouping of the various errors.

Imperfect centering of the apparatus and friction in the bearings are particularly important sources of error here. For the *pendulum experiment* of Foucault and Gaufs, all sources of error have been examined quantitatively in an exemplary manner by K a m e r l i n g h O n n e s;*)[263] for the Foucault *top experiment*, in contrast, such an examination has never been undertaken.

The great historical significance of the Foucault top experiment thus appears to us to lie less in the proof of the Earth's rotation than in the fact that this experiment attracted general attention to the top-effect, and that the awareness of the top-effect was essentially advanced by the brilliant, formula-free immediateness of the Foucault conception.[264]

Before we examine the theory of this experiment critically, we first wish to report in more detail on the second experimental disposition of Foucault, the *top with two degrees of freedom.* The axis of the rotor is now no longer fixed in space; *the axis rather seeks, according to Foucault, to place itself as parallel to the axis of rotation of the Earth as the particular circumstances of the experiment permit.* Foucault thus speaks of the *tendency of the rotation axes to parallelism*;**) parallelism of the axes is understood not only as coincidence of the direction of the axes,

*) Dissertation Groningen 1879. Nieuve bewijzen voor de aswenteling der aarde.

**) Sur la tendance des rotations au parallélisme. Comptes Rendus l. c. p. 602.

but rather, at the same time, coincidence of the rotation sense—one can thus speak, more precisely, of the *tendency to equi-orientational or homologous parallelism.*

At approximately the same time as Foucault, G. Sire*) made the same law the subject of a communication to the Paris Academy, and applied it to the proof of the rotation of the Earth, without carrying out an experiment himself. The theoretical deliberations of Sire on which the law is based, however, are not unobjectionable, since they suffer a certain ambiguity of the word axis (figure axis, rotation axis, impulse axis). A somewhat similar objection may be made against the brilliantly written works of Foucault; these works are intended, because of their brevity, to be more of a descriptive than a demonstrative nature. (Cf. our critique of the popular top literature in Chap. V, §3 sub 2.)

Foucault attempted to use the named law as a basis for the behavior of the rotor axis in the two special cases in which the rotor axis was free to move only in the horizontal plane or only in the vertical plane through the meridian of the observation location. The restriction of the motion is accomplished in both cases by clamping the inner ring at a right angle with respect to the outer ring. In the first case, the rotation axis of the outer ring was placed in the direction of the plumb line; in the second case, the rotation axis of the outer ring was placed perpendicular to the meridional plane of the observation location.

In the first case, where the axis of the rotor cannot leave the horizontal plane, an actual parallelism between it and the axis of the Earth is not possible; the axis of the rotor then strives to the direction that forms the smallest angle with the axis of the Earth; that is, the direction of the meridian. And indeed, the side of the axis from which the rotor is seen to turn in the counterclockwise direction points to the north, since the Earth rotates about the north pole in the same sense. *Our horizontally mobile rotor thus behaves similarly to the magnetic needle in a declination compass* (naturally with the difference that in Foucault's experiment the astronomical meridian takes the place of the magnetic meridian). In association with this analogy, we can denote the side of the axis from which the rotor is seen to turn in the counterclockwise sense as the *north pole* of the rotor, and the opposite side as the *south pole.*

In the second case, in which the axis of the rotor is free to move in

*) A later publication of Sire is found in the Bibliothèque universelle de Genève, Arch. d. scienc. phys. et natur., t. 1 (1858), p. 105.

the meridional plane, exact parallelism of the rotor axis with the axis of the Earth is not only sought, but rather (for a sufficiently long maintenance of the rotor rotation) is achieved. The axis of the rotor moves, if it is initially horizontal, in such a manner that, in the northern hemisphere, its "north pole" rises up out of the horizontal plane, and the line that connects the north and south poles of the rotor is directed parallel to the line that connects the north and south poles of the Earth. *Our rotor axis that moves in the meridional plane can thus be compared with the magnetic needle in an inclination compass* (naturally again with the difference that the exact geographical latitude lines take the place of the quite irregular lines of equal inclination on the surface of the Earth). An essential difference is *that the "north pole" of the rotor ascends in the northern hemisphere, while the north pole of the inclination needle descends.*[265]

The theoretical possibility thus exists, as Foucault emphasizes, of deriving the position of the meridian and the position of the Earth's axis at an arbitrary location merely from observations of the top, without astronomical or magnetic observations. It is obvious that the approach of the axis of the top to the meridian or to the direction of the Earth's axis must occur not monotonically, but rather with oscillations. The turning-force that leads the horizontally mobile axis of the top to the meridian, for example, produces a certain angular acceleration and angular velocity about the vertical axis. While the turning-force vanishes for the meridional position of the axis of the top, the angular velocity does not vanish at the same time. This leads the axis of the top across the equilibrium position, so that the restoring force is reversed in sense, and acts to first decelerate and then accelerate in the reversed sense. The axis of the top must therefore oscillate about the meridian—likewise in analogy with the magnetic needle. If the axis of the top is initially in the direction of the meridian, but so that its "north pole" points to the south, then this position is also an equilibrium position, since the turning-force on the axis of the top vanishes, but is evidently unstable: for a small deviation from this position, the turning-force strives to make the deviation larger, and the north end of the axis turns over to the north.

Foucault disputed that the position of the meridian or the location of the Earth's axis could be determined with sufficient accuracy in the

prescribed manner. It appears that Foucault also examined the second experiment more for its general possibility than for its exact implementability. —

We mention further that Foucault coined, in association with his experiments, the now commonly used word *gyroscope.* This word expresses in a forceful way the result of the Foucault experiment; *namely, that the top is a means to make an existing rotational motion* (or gyration) *recognizable*, just as the electroscope denotes a means for making the presence of electrical charges visible. If it were also possible to establish the magnitude of an existing rotational motion through a quantitative measurement of the motion of the top, then one might even bestow to the top the more far-reaching designation of a "gyrometer."

On the other hand, it appears to us inappropriate to generalize the designation "gyroscope," and to use it as an equivalent to the word top, as is frequently done in the literature. In fact, the designation gyroscope expresses only one particular application of the multifaceted interest and importance of the concept of the top, and there is no reason to abandon the characteristic designation *top* (*turbo, toupie, Kreisel*). —

We must now deepen the theory of the Foucault experiments; we begin with the top of three degrees of freedom. It is far from our desire to accompany this experiment with extensive analytic developments from the theory of relative motion,*) developments whose final result, according to their fundamental assumptions, can be nothing other than the confirmation of Foucault's statement that the axis of the top essentially retains its position in absolute space. The difficulty and obscureness of these developments have their basis only in the fact that it is not always assumed that the gyroscope is vanishingly small with respect to the Earth, or that its rotational velocity is infinitely large compared to

*) These developments are treated, for example, in the works of Quet (Liouville's Journal t. 18 (1853)), Lottner (Crelle's Journal Bd. 54 (1857)), and Bour (Louiville's Journal (2) t. 8 (1863)).[266] A summary is given by Gilbert: Étude historique et critique sur le problème de la rotation (from the Annales de la Société Scientifique de Bruxelles, t. 2 (1878)). We further mention the great work of Gilbert: Mémoire sur l'application de la méthode de Lagrange à divers problèmes du mouvement relatif (ibidem, t. 6 and 7, 1881–1883) and the work of Budde: Allgemeine Mechanik der Punkte und starren Systeme (Berlin 1890, 1891, Bd. 2, Nr. 294).

that of the Earth. We refer for this matter to the relevant critique of E. G u y o u,*) to which we would add only that one may neglect the inertial effect of the inner and outer rings compared to that of the rotor with the same enormous degree of approximation with which one neglects the rotational velocity of the Earth compared to that of the top, as we will discuss below.[267]

We first wish to assume explicitly that it may be permitted to consider the outer and inner rings as *massless*, and *to disregard the effects of friction.* We then have to speak of the rotor alone. The rotor is free to move about its center of gravity and is free of external forces, since the gravitational force applied at the support point does not come into consideration. The motion of the rotor thus consists, generally speaking, of a *regular precession* with respect to absolute space. The motion of the center of gravity due to the rotation of the Earth in no way influences this rotational motion. For the motion of the center of gravity and the rotation about the center of gravity are, as is well known, two processes that are superposed smoothly in the absence of external forces, without disturbing one another in any way.

This would also hold if the center of gravity were not led approximately uniformly in a straight line, as it in fact is by the rotation of the Earth in the time duration of a few minutes, but rather *if the center of gravity were led in an arbitrary manner along a path with an arbitrarily sharp curvature.* Indeed, it would be valid not only for a rapid spin of the rotor, but rather equally well *for a nonspinning rotor*, always under the assumption of the frictionlessness of the guidance and the masslessness of the rings. In fact, the force-free motion of the symmetric top is a regular precession for an arbitrary magnitude of the eigenrotation; the greater or lesser eigenrotation that is imparted to the top merely determines whether the resulting precession cone has, for a given lateral impact, a smaller or larger opening angle. Were it attained that the axis of the rotor stood instantaneously at rest in absolute space at the beginning of the experiment, then its direction with respect to absolute space would be precisely retained, and the opening angle of the precession cone would be and remain exactly zero, completely independently of whether the rotor

*) Sur une solution élémentaire du problème du gyroscope de Foucault, Comptes Rendus, t. 106, Paris 1888, p. 1143.

turned or not and whether the center of gravity of the apparatus moved or not; for after we have assumed away friction, there is nothing that can cause the initially stationary axis of the rotor to rotate. We would thus have the stability of the axis of the top that is claimed by Foucault, without the large eigenrotation that is regarded by Foucault as an indispensable means.

The just-assumed initial state of the axis of the top is not, however, to be achieved experimentally. The experimenter can judge the initial state of rest of the axis of the top only from the standpoint of the moving Earth; he seeks to effect not absolute rest in space, but rather rest relative to the Earth. We now assume that the latter is precisely achieved, and consider the form of motion of the axis of the top if, in particular, the top is not spun.

In the initial state of the Foucault experiment, the axis of the top is horizontal; the angle between the axis of the top and the rotation axis of the Earth is α, which is contained between φ and $\pi - \varphi$ (φ = the geographical latitude of the location), and depends on the azimuth of the axis of the top with respect to the meridian. The initial state of velocity is a rotation about the axis of the Earth with magnitude ω (ω = the rotational velocity of the Earth). This rotation is decomposed into a component $\omega \cos\alpha$ about the figure axis and a component $\omega \sin\alpha$ about an equatorial axis of the rotor. The initial impulse of the top has, with respect to these same axes, the components $C\omega\cos\alpha$, $A\omega\sin\alpha$; it forms, in the case of the rotor (oblate ellipsoid of inertia), an angle $\beta < \alpha$ with the figure axis, which angle is determined by

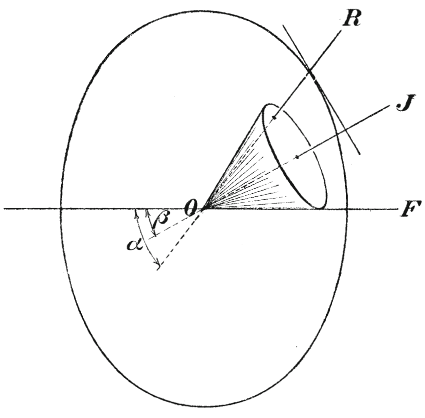

Fig. 110.

$$(1) \qquad \operatorname{tg}\beta = \frac{A}{C}\operatorname{tg}\alpha.$$

The angle β can also be found by a well-known construction (cf. page 106) that is indicated in Fig. 110. The direction of the impulse axis OJ follows from the direction of the rotation axis OR (which here coincides with the axis of the Earth) and the direction of the figure axis OF, in that one attaches to the trace of the ellipsoid

of inertia in the plane ROF the tangent at the intersection point with OR, and drops the perpendicular to this tangent though O.

In the resulting motion, the figure axis describes a precession cone in space about the impulse axis with the just determined opening angle β. In contrast, a direction emanating from O that is fixed to the Earth describes, due to the rotation of the Earth, a cone about the rotation axis of the Earth. From the difference between the cones, it follows that the nonrotating top, judged from the Earth, would move, and would thus (for completely eliminated friction) function as a gyroscope in the sense of Foucault.

The assumption that the top is exactly at rest with respect to the Earth in its initial state is naturally also impermissible. Even in the most careful experiments, the axis of the top will have an initial angular velocity with respect to the Earth, which, with consideration of the smallness of the Earth's rotation, can very possibly be greater than the angular velocity corresponding to the latter. The initial impulse vector, which is determined by the composition of this angular velocity with the angular velocity of the Earth's rotation, can then have any arbitrary position, and the precession cone that the figure axis describes about this impulse vector can have any arbitrary opening angle. If, for example, the initial impact directly canceled the component of the Earth's rotation with respect to the figure axis of the top, then the eigenrotation of the top becomes accidentally zero, and the impulse vector would fall onto an equatorial axis; the precession cone would then degenerate into the plane normal to this axis. If, on the other hand, the component of the Earth's rotation with respect to the equatorial plane of the top is accidentally canceled by the initial impact, then the precession cone would become infinitely narrow and would coincide with the figure axis; the figure axis itself would then stand absolutely still in space. With consideration of such an uncontrollable small initial impulse, the further motion of the top would therefore become completely uncertain.

And one can now avoid this uncertainty if one imparts to the top an eigenrotation that is large with respect to the rotation of the Earth. A velocity of one rotation per second would already suffice, since this is $24 \cdot 60 \cdot 60$ times the velocity of the Earth's rotation. Foucault, in fact, worked with approximately this rotational velocity.*) The total impulse of the top, which is composed from this intentional

*) Cf. here the Instructions sur les expériences du gyroscope in the book Recueil des travaux scientifiques de L. Foucault, Paris 1878, p. 417.

eigenrotation, the unavoidable initial impulse, and the impulse of the Earth's rotation, will then have nearly the direction of the figure axis. At the same time, the precession cone will be so narrow that we can speak, with a precision sufficient for all experiments, of the absolute rest of the figure axis in space.

The purpose of the eigenrotation that is imparted to the top is thus to achieve, in the first place, *that the top will be free of the uncontrollable impulse of the initial actuation.* A completely determinable and well-defined motion of the top is then made possible, and *the precession cone will be made sufficiently narrow*, so that this motion will be made as simple as possible; it is perceptibly transformed, namely, into a simple rotation about the figure axis that is fixed in space. If Ω is the eigenrotation of the top, ω, as previously, the rotational velocity of the Earth, α the initial angle of the figure axis with respect to the rotation axis of the Earth, ω_0 the angular velocity relative to the Earth imparted to the rotor by unintentional impacts, and γ the angle that the figure axis forms with the axis of ω_0, then the opening angle β of the precession cone is determined, similarly to formula (1), by

$$\operatorname{tg}\beta = \frac{A}{C}\cdot\frac{\omega\sin\alpha + \omega_0\sin\gamma}{\Omega + \omega\cos\alpha + \omega_0\cos\gamma}. \tag{2}$$

If Ω is large compared to ω and ω_0, then one has, perceptibly, $\beta = 0$. If we assume, as was done above, that ω_0 and ω are of the same order of magnitude, then we can say concisely that the opening angle of the cone will be of order ω/Ω, and that the direction of the figure axis is to be regarded as invariable if and only if one neglects quantities of order ω/Ω.

Thus the Foucault result of the stationarity of the rotor in space is confirmed as sufficiently exact under the omissions made thus far, and, at the same time, the proper role of the eigenrotation is revealed more clearly than in Foucault.

The influence of the *mass system of the suspension rings* should be studied next, or it should be demonstrated that they are without perceptible influence on the position of the rotor. As long as we assume that the suspension rings are massless, the axis of the rotor stands perceptibly still in space for a sufficient eigenrotation. Correspondingly, a diameter D of the inner ring (namely, that which coincides with the axis of the rotor) will, under this assumption, remain fixed in space. On the other hand, a diameter D' of the outer ring (namely, the diameter that

falls along the plumb line of the observation location) will be constrained by its attachment to the rotating Earth to move in a completely determined manner. It is clear, however, that the motion of the system of our two rings is completely established by the motion of the two diameters D and D'; if the diameter D actually stands exactly still in space and the diameter D' exactly participates in the motion of the plumb line, then the motion of our two rings is *necessarily* determined. Their angular velocity will evidently be *of the order of magnitude of the angular velocity of the rotation of the Earth.* In more detail, we saw on page 732 that as long as the rotor deviates relative to the Earth only slightly from its horizontal initial position, the angular velocity of the outer ring will be equal to $\omega \sin \varphi$; that of the inner ring, as likewise follows from the previous deliberation, will be equal to $\omega \cos \varphi \sin \lambda$, where λ signifies the azimuth of the axis of the top with respect to the meridian of the observation location. Since the axis of the rotor subsequently maintains its position in space and is thus removed in the course of time from the horizontal of the observation location, these values of angular velocity continually change, but they always remain of the order of magnitude ω.

We now imagine that the described motion also occurs for nonvanishing masses of the rings. It is thus necessary that the rings be imparted at the beginning of the motion with the impulses $A_1 \omega \cos \varphi \sin \lambda$, $A_2 \omega \sin \varphi$, where A_1 and A_2 signify the moments of inertia of the inner and outer rings about one of their diameters, and that this impulse be changed in the manner that corresponds to the variability of the angular velocity. The impulse remains of the order of magnitude of $A_1 \omega$ and $A_2 \omega$. If we compose this impulse with the impulse of the eigenrotation of the rotor, which amounts in essence to $C\Omega$, there results a total vector that always deviates in direction and magnitude only slightly from the constant eigenimpulse of the rotor. The direction difference as well as the relative magnitude difference of the two vectors is, if we set the order of magnitude of the ratios of the moments of inertia A_1/C and A_2/C equal to 1 and thus calculate unfavorably, of the order of magnitude ω/Ω. (For the actual implementation of the Foucault gyroscope, the ratios A_1/C and A_2/C are indeed essentially smaller than 1.)

We can thus say that for our supposed enforced motion of the ring masses, as it would be determined by the stationarity of the rotor axis, there corresponds a total impulse that we may regard as constant in

magnitude and direction, in so far as we neglect direction and magnitude changes of the order of ω/Ω. The total impulse therefore remains constant in the same sense and with the same degree of precision as the figure axis retains its position in space with the disregard of the ring masses. In fact, the invariability of the rotor axis was also only approximate, as was emphasized in association with equation (2), and occurs only with the neglect of quantities of order ω/Ω.

We thus conclude, in that we now go over from the previously considered enforced motion to the free motion of the mass system of the rotor and the inner and outer rings, that the free motion will be identical with the enforced, for the same choice of the initial state, in so far as we neglect differences of the order ω/Ω. For it is required in the free motion that the total impulse of the mass system remain constant in magnitude and direction. The considered enforced motion satisfies this requirement within the precision bound ω/Ω. The enforced motion thus coincides, within the same precision bound, with the natural motion of the mass system.

In other words, *the influence of the masses of the Cardanic suspension on the motion of the Foucault gyroscope with three degrees of freedom is only of the order of magnitude ω/Ω, and can in no way be detected in the observation.* It not only may, but rather must be neglected in a consistent manner, if one at all wishes to speak with Foucault of the invariability of the axis of the rotor. —

In order to avoid the occurrence of misunderstandings, we wish to emphasize explicitly that the angular change of the outer ring with respect to the Earth (or the horizontal component of the relative motion of the axis of the rotor) that is to be measured according to the experimental method of Foucault is not, in its turn, of the order of magnitude that is neglected here. This angular change amounts, namely, to $\omega \sin\varphi\, \Delta t$. Its ratio with respect to quantities of order ω/Ω is $\Omega \sin\varphi\, \Delta t$. Here $\Omega\Delta t$ signifies the rotation angle of the rotor during the observation time, and is thus an extraordinarily large multiple of 2π for a moderately rapid rotation and a typical observation time of 8 minutes. We thus recognize that the value of the gyroscopic effect to be observed is in no way obscured by the neglect of order ω/Ω. —

Friction may have a disproportionally larger influence than the masses of the suspension rings. We consider in part the friction in the guide bearings of the axis of the outer ring, and in part the resistance

that is developed between the knife edges of the inner ring and their bearing surfaces on the outer ring. The investigation of this source of error, to which air resistance, air disturbances, the warming of the materials, etc. are still to be added, would certainly be more important for the actual understanding of the Foucault experiment than the unnecessarily general and mathematical considerations of relative motion that were mentioned on page 736.

We can illustrate the influence of bearing friction in the grossest and roughest measure by a simple experiment. We consider a top whose center of gravity lies at the center of a Cardanic suspension (Fig. 2). We give the rotor a strong rotation and place its axis in an initially horizontal position. We then turn the frame slowly about the vertical. If the circumstances are favorable (that is, the friction in the bearings is small, the eigenrotation is strong, and the turning of the frame is slow), the axis of the rotor seems at first to retain its initial position, and the plane of the outer ring thus remains fixed in space. This result, however, is only the consequence of an imprecise observation. For a longer maintenance of the turning of the frame, or for intentionally increased friction in the bearings, we see that the axis of the rotor is slowly uprighted and that the inner ring thus tilts, while the outer ring apparently continues to retain, in essence, its initial position. If, on the other hand, we turn the frame about the horizontal axis of the inner ring, then we again notice that the axis of the rotor seems at first to remain at rest, so that the inner ring is stationary in its original horizontal plane. If the rotation of the frame in maintained longer or if the bearing friction is intentionally increased, however, we see that the axis of the rotor deviates laterally in the horizontal plane, so that the plane of the outer ring rotates about its axis.

The basis for these motions is obviously friction. If we turn the frame about the vertical, then the bearings of the outer ring move relative to its pintles, which are approximately held fixed by the rotor, and there results a frictional moment about the axis of the *outer* ring; this first sets into motion, as we see in the experiment, not the outer, but rather the *inner* ring. If, however, we turn the frame about the previously named horizontal axis, then the bearings of the inner ring slide with respect to its pintles, and a frictional moment thus appears about the axis of the *inner* ring; this moment sets into motion not the inner, but rather primarily the *outer* ring.

The explanation of these initially paradoxical phenomena may be taken, at least qualitatively, from the theory of the heavy top. Under the influence of a frictional moment about the rotation axis of the *inner* ring, our rotor with its support point at the center of gravity behaves, mutatis mutandis, like a heavy top. For the named frictional moment has, just as gravity has for the noncoincidence of the center of gravity and the support point, the horizontal line perpendicular to the figure axis (the "line of nodes") as its axis. The consequence is a pseudoregular precession of the rotor, in which the figure axis of the rotor deflects in the horizontal plane. The plane of the inner ring thus remains in the mean horizontal, and the plane of the *outer* ring will turn. The deliberation can also be carried over in a corresponding manner to the initially considered turning of the *outer* ring about the vertical axis, and then gives a precession of the rotor in a vertical plane, and therefore a rotation of the *inner* ring. We will return to the latter case in the following chapter, where we will give a thorough theory of the relevant phenomenon for the torpedo directional guidance apparatus.

These results carry over to the Foucault experiment as follows. What for us is the frame of the top, is for Foucault the Earth. Its rotation occurs about the polar axis. We decompose this rotation into its three components with respect to the plumb line (that is, the rotation axis of the outer ring), the originally horizontal rotation axis of the inner ring, and the figure axis of the rotor. The frictional resistances that correspond to the first two rotation components act on the rotor in the manner of our experiment; one component turns the inner ring and thus deflects the axis of the rotor in the vertical plane, and the other component turns the outer ring and thus effects a horizontal deflection of the rotor axis. Both circumstances perturb the apparent motion that the spatially fixed rotor should describe, according to Foucault, with respect to the Earth. The third rotation component with respect to the figure axis does not come into consideration; the corresponding frictional moment adds to the friction generated by the eigenrotation of the rotor axis in its bearings, and is to be neglected in comparison.

There are thus various frictional influences in the Foucault experiment that perturb the absolute rest of the axis of the rotor.

There now arises the question of how one can master these frictional influences. The means is again provided by a *sufficiently high eigenrotation of the rotor.* (Foucault himself leaves us somewhat in doubt concerning the role of the eigenrotation in his experiments, as was

already mentioned above.) With respect to the initial motion of the rotor axis, we saw that the eigenrotation has the result *of making the precession cone of this axis sufficiently narrow.* With respect to friction, on the other hand, we must say that the eigenrotation has the purpose *of making the precessional velocity corresponding to the various frictional influences as small as possible.* The precessional motion considered now is completely different from the previous. In the precessional motion effected by friction that we derive by analogy with the heavy top, the axis of the rotor describes a degenerate cone (or a fan) in a plane, and indeed such a cone in the horizontal or the vertical plane, according to whether we consider the frictional moment about the rotation axis of the inner ring alone or the outer ring alone. (In actuality, the motions will naturally be superposed, and there is also to be added the minimal oscillation due to the previously considered precessional motion.) Since the precessional velocity of the heavy top is equal to P/N, where P signifies the moment of gravity and N the eigenimpulse of the top [cf., for example, page 305, equation (13)], the velocity of the precession effected by the bearing friction is equal, by analogy, to M/N, where M signifies one or the other frictional moment, and N is again the eigenimpulse. Through the enlargement of N, one can, in any case, make this precessional velocity so small that the axis of the top does not deviate perceptibly from its initial position in space during an observation time of a few minutes. Indeed, we see that if it is at all permitted to disregard friction, it is permitted only for a short time duration and a sufficiently large eigenimpulse. *Even if the eigenimpulse of the frictional moment cannot be eliminated, its effect can be delayed, so that it becomes inessential for not too long an observation time.*

It is not possible, however, to determine theoretically how large the eigenimpulse must be in order to achieve this, since this depends on the magnitude of the frictional moment M, and therefore on the construction of the bearings and knife edges. Here a precise experimental investigation of the error sources must be made, which appears to be wanting in Foucault's own work. The experimental genius of Foucault guarantees us that the frictional effect M for his apparatus was very small; how small it was, however, we cannot judge from his communications.

Another difficulty of the first Foucault experiment, the necessity of a very precise centering of the rotor,*) is avoided through a happy modification of the gyroscope, the so-called *barogyroscope*, which will be discussed below.

We next go over to the second Foucault experiment (the top with two degrees of freedom), and must prove here the two interesting theorems that a) a rotor that may move in the horizontal plane behaves like a declination needle, and b) a rotor that may move in the meridional plane behaves, mutatis mutandis, like an inclination needle.

The proof of both these theorems is immediately illuminated if we rely on the previously developed concept of the deviation resistance (cf. Chap. III, §6); detailed analytic developments, as they are given for this purpose by Gilbert,**) appear just as out of place here as in the previous case. The following simple considerations coincide in result with the Gilbert developments.

a) *Rotation axis of the outer ring along the plumb line, inner ring clamped at a right angle with respect to the outer ring, axis of the rotor sweeping the horizontal plane.* We decompose the Earth's rotation ω into its two components with respect to the plumb line and the meridian of the observation location. The first component is $\omega \sin\varphi$, where φ is the geographic latitude. This component does not influence, for sufficiently small friction in the pintles of the outer ring, the absolute position of the rotor; the rotor simply does not partake of this rotation, so that it naturally also prevents the inner and outer rings from following this rotation; the rotor behaves with respect to this component just as a top with three degrees of freedom behaves with respect to the entire rotation of the Earth. The second component is $\omega \cos\varphi$. If we imagine the position of the rotor in the horizontal plane as fixed for an instant, this component would lead the axis of the rotor on a circular cone about the meridian, and the top would describe a regular precession. Due to its inertia, the top resists this enforced motion with a moment whose axis is simultaneously perpendicular to the figure axis and the axis of the precession cone, and therefore falls, in our case, along the plumb line. According to page 175, equation (1), the magnitude of this moment is, if we insert for the precessional velocity there denoted

*) Cf. the Instructions sur les expériences du gyroscope cited above.

**) Cf. §XV and XVI of the work cited on p. 736: Mémoire sur l'application etc.

by ν the value $\omega\cos\varphi$, and introduce the eigenimpulse N of the rotor*) into the formula,

$$K = -\omega\cos\varphi\sin\vartheta\,(N - A\omega\cos\varphi\cos\vartheta). \tag{3}$$

Here ϑ is the angle between the figure axis of the top and the axis of the precession cone; that is, in our case, the angle between the figure axis and the meridian. For the determination of the sign, it is stipulated that we wish to measure ϑ from the northern side of the meridian, and that we reckon the (positive) figure axis as the side of the rotor axis about which the eigenrotation follows in the same sense as the rotation of the Earth about the line that connects the center of the Earth and the North Pole, and that we therefore imagine, with use of the manner of expression introduced on page 734, that the figure axis is drawn from the midpoint of the rotor to its "north pole." The product ωN in equation (3) is, on the basis of this stipulation, a *positive* quantity.

Moreover, the second term in the parentheses of (3) is unquestionably to be neglected with respect to the first. The order of magnitude of the second term is related to that of the first, namely, as $A\omega$ to N, or (with disregard of the difference between the equatorial and polar moments of inertia), as the velocity of the rotation of the Earth to the angular velocity of the rotor, or as the duration of one rotor rotation to the length of the day. We thus write, instead of (3),

$$K = -N\omega\cos\varphi\sin\vartheta. \tag{3'}$$

Should the assumed precessional motion of the rotor now be sustained under the unchanging inclination ϑ with respect to the meridian, a moment $-K$ about the plumb line that overcomes the inertial resistance K must be exerted. If this moment is not exerted, the rotor moves as if a moment $+K$ acted about the plumb line; this moment changes the angle ϑ. The plumb line is an equatorial principal axis for the rotor and for the outer ring; for the inner ring, in contrast, the plumb line is the figure axis itself. If we denote by A_1, C_1, A_2, C_2 the respective equatorial and polar principal moments of inertia of the inner and outer rings, then the sum of the relevant moments of inertia of the rotor and the

*) The eigenimpulse N is expressed (cf., for example, p. 222 above) in terms of the Euler angles φ, ψ, ϑ as $N = C(\varphi' + \cos\vartheta\,\psi')$; the angular velocities φ' and ψ' are, however, denoted in equation (1) of page 175 by μ and ν. The eigenimpulse is thus $N = C(\mu + \cos\vartheta\cdot\nu)$. The Euler angle φ obviously has nothing to do with the geographic latitude φ used in the text.

inner and outer rings about the plumb line will be $A + C_1 + A_2$. The equation of motion thus becomes

$$(A + C_1 + A_2)\vartheta'' = K = -N\omega \cos\varphi \sin\vartheta. \tag{4}$$

It is self-evident that the eigenimpulse of the rotor will not be simultaneously changed by the rotation of the Earth, since the axis of K stands perpendicular to the figure axis. Thus the quantity N in the preceding equation is a constant, which we can regard as the second equation that is necessary, in addition to (4), for the complete description of the motion of our top.[268]

Equation (4) now shows immediately that *the axis of the rotor is in equilibrium only if it is in the direction of the meridian.* For we have $\vartheta'' = 0$ only if $\vartheta = 0$ or $\vartheta = \pi$. *Of the two equilibrium positions $\vartheta = 0$ and $\vartheta = \pi$, the first is stable, and the second is labile.* Due to the sign of the right-hand side of (4), the axis of the rotor will, for a disturbance of the equilibrium position $\vartheta = 0$, be led back to this position by the resulting angular acceleration; for the equilibrium position $\vartheta = \pi$, in contrast, the axis of the rotor is removed farther from this position when it is disturbed. *In the stable equilibrium position, the eigenrotation of the rotor is in equi-orientational parallelism with the meridional component of the rotation of the Earth.* For we wished, in order to make the sign of ωN positive, to determine the angle ϑ between the meridian and the figure axis so that the eigenrotation about the figure axis followed in the same sense as the rotation of the Earth about its axis, or as the meridional component of the same about the northern half of the meridian from which we measured the angle ϑ. *The tendency to equi-orientational parallelism of the rotation axes that was emphasized by Foucault is shown in the appearance of the acceleration that directs the axis of the rotor toward the stable equilibrium position.*

The motion in question may be described most simply and completely if we compare it with the motion of a mathematical pendulum. In fact, (4) is nothing other than the usual differential equation of pendulum motion. We can write the latter, if we understand by l the length of the pendulum and by ϑ the angle that l encloses with the stable equilibrium position (that is, with the instantaneous direction of gravity), as

$$\vartheta'' = -\frac{g}{l}\sin\vartheta. \tag{4'}$$

In order to transform equations (4) and (4′) into one another, it is necessary only to choose l as

$$l = \frac{g(A + C_1 + A_2)}{N\omega \cos\varphi}. \tag{5}$$

This formula gives the length of the corresponding mathematical pendulum whose motion, for equal initial values of ϑ and ϑ', is exactly identical with the motion of our rotor. The length and the oscillation period of the pendulum will always be smaller as N is larger; correspondingly, the directional force of the Earth's rotation on our rotor increases with the magnitude of the eigenimpulse N.

The comparison with the declination needle may now be drawn in the most simple manner. Since the equation of motion of such a needle is $J\vartheta'' = -MH \sin\vartheta$, where J signifies the moment of inertia of the needle, M is the magnetic moment of the needle, and H is the horizontal component of the Earth's magnetic field, the length of the mathematical pendulum that corresponds to this magnetic needle is

$$l = \frac{gJ}{MH}. \tag{5'}$$

If one equates the pendulum lengths given in (5) and (5′), one recognizes how Foucault's identification of the spinning rotor with the magnetic needle may be realized quantitatively. If one imagines, for example, that the moment of inertia J of the needle coincides with the total moment of inertia $A+C_1+A_2$ that comes into consideration for the top apparatus, then one must simply choose the eigenimpulse of the rotor so that $N\omega\cos\varphi = MH$; *the motion of our rotor with eigenimpulse N will then be a congruent image, for equal initial values of ϑ and ϑ', with the motion of a magnetic needle with magnetic moment M.*

b) *Rotation axis of the outer ring perpendicular to the meridional plane, inner ring fixed perpendicular to the outer ring, rotor axis mobile in the meridional plane.*

Here we must refrain from the decomposition of the Earth's rotation into components, since the total rotation ω influences the position of the rotor in the meridional plane. If we imagine the angle ϑ between the axis of the rotor and the axis of the Earth's rotation as instantaneously fixed, then the rotor would describe a regular precession about the axis of the Earth. It would resist this precession, due to its inertia, with a moment K whose axis is simultaneously perpendicular to the axis of the rotor and that of the Earth's rotation, and therefore in the normal to the meridional plane; that is, along the rotation axis of the outer ring. According to equation (1) of page 175, the magnitude of the moment K is,

if we now insert ω in place of ν and N in place of $C(\mu + \nu \cos\vartheta)$,

$$K = -\omega \sin\vartheta (N - A\omega \cos\vartheta). \tag{6}$$

Just as in a), we neglect the second term in the parentheses with respect to the first, so that (6) becomes more simply

$$K = -N\omega \sin\vartheta. \tag{6$'$}$$

If we stipulate, as in a), that we draw the figure axis from the midpoint of the rotor toward the side seen from which the rotation of the rotor occurs in the same sense as the rotation of the Earth about the North Pole, and we measure the angle ϑ from the northern half of the Earth axis to the so-defined figure axis, then the product $N\omega$ in the preceding equation will be positive.[269]

Should the rotor retain its position in the rotating meridional plane, a moment $-K$ about the rotation axis of the outer ring is necessary to overcome the inertial resistance of the rotor. If such a moment is not exerted, the angle ϑ between the Earth's axis and the figure axis must change, in such a measure that the product of the moment of inertia of the moving components and the angular acceleration is equal to K. This leads, just as above, to the differential equation

$$(A + C_1 + A_2)\vartheta'' = K = -N\omega \sin\vartheta. \tag{7}$$

From this equation, the conclusion immediately follows that *the rotor is in equilibrium in the meridional plane only if it has the direction of the Earth's axis; that is, only in the two positions $\vartheta = 0$ and $\vartheta = \pi$. The former position is a stable equilibrium position, and the latter is labile. In that the acceleration of the rotor determined by* (7) *leads toward the stable equilibrium position, it strives to direct the axis of the rotor parallel to the axis of the Earth in a homologous sense.*

The present motion is also congruent with the motion of a simple pendulum. *The corresponding pendulum length is now*

$$l = \frac{g(A + C_1 + A_2)}{N\omega}. \tag{8}$$

On the other hand, the motion of the inclination needle can also be identified with the pendulum motion. If J and M signify the moment of inertia and the magnetic moment of the needle and T is the total

intensity of the Earth's magnetic field, then the length of the pendulum that corresponds to this inclination needle is

$$l = \frac{gJ}{MT}. \tag{8'}$$

From the comparison of (8) *and* (8′), *one recognizes how the behavior of our rotor axis that moves in the meridional plane may be identified quantitatively with the behavior of the magnetic needle in the inclination compass*, where, however, the difference between the two motions that was emphasized on page 735 is to be kept in mind.

c) The preceding results may be summarized and generalized if one assumes that the axis of the rotor moves in a *plane E that is oriented arbitrarily with respect to the Earth.* In order to achieve this arbitrary orientation, one can again fix the plane of the inner ring at a right angle with respect to that of the outer ring, and then place the rotation axis of the outer ring relative to the Earth so that it stands perpendicular to the plane E. If λ signifies the angle between the axis of the Earth's rotation and the plane E, then the effective component of the Earth's rotation is $\omega \cos \lambda$, and the length of the corresponding pendulum follows as

$$l = \frac{g(A + C_1 + A_2)}{\omega N \cos \lambda}. \tag{9}$$

This formula is transformed in cases a) and b), where, in particular, $\lambda = \varphi$ and $\lambda = 0$, into equations (5) and (8), respectively; it is due to Gilbert.*)

Two remarks may be added on the influence of friction and on the influence of the masses of the suspension system; these remarks refer at the same time to cases a) and b), as well as to the generalized case c).

Friction in the bearings of the outer ring will naturally also be perceptible for the gyroscope with two degrees of freedom. While the consideration above shows that the oscillation amplitude of the rotor axis must remain constant if friction is neglected, the amplitude will generally be damped by friction. Thus the rotor axis will approach the stable equilibrium position with an oscillation of decreasing amplitude, and will finally come to rest in this position if the eigenrotation of the rotor is

*) Equation (130) in the previously cited work Mémoire sur l'application etc. We differ in the text from Gilbert only in that we have suppressed a term with the factor ω^2 in the passage from (3) to (3′) and from (6) to (6′). With consideration of the degree of precision of the entirely theoretical assumptions, this term can have no significance.

maintained for a sufficiently long time. With respect to the quantitative relations, we may simply call upon the analogy with the simple pendulum or the magnetic needle. For a similar construction of the bearings, the effect of friction for the considered experiment will be similar to that for the oscillations of the simple pendulum and the magnetic needle, which, in their turn, are naturally always damped oscillations.

For what concerns the effects of the masses of the outer and inner rings, it may be surprising that we have expressed these effects in our last formulas, while we said in the discussion of the first Foucault experiment that they were to be neglected. This is explained by the fact that in the first Foucault experiment (with the neglect of friction) the rotor axis was perceptibly fixed in space and the angular velocities of the ring masses are only of the order of the rotation of the Earth, while in the second Foucault experiment, in contrast, the rotor axis experiences actual accelerations in which the masses of the inner and outer rings must take part. While in the first Foucault experiment the influence of the ring masses on the observed magnitude of the relative motion of the rotor is vanishingly small (of order ω/Ω; cf. page 742), this influence in the second Foucault experiment is of the same order of magnitude as the observed motion of the rotor itself, so that, for example, the moments of inertia C_1 and A_2 of the rings are to be added directly to the moment of inertia A of the rotor in formula (9). —

It remains, finally, to speak of a purposeful modification of the Foucault gyroscope, the previously named *barogyroscope* of Gilbert. As the name implies, both gravity and the rotation of the Earth come into play for this instrument. The device is illustrated in Fig. 111;*) we describe it by drawing the comparison with the Foucault gyroscope.

One sees in the figure the rotor D with its axis a, on which is found at E a gear for the actuation of the rotor and a sliding weight p on the lower elongation of the axis. We can designate the frame C as the inner ring; it rests at A and A' on knife edges. We can compare the bracket S with the outer ring of Foucault. It may rotate in the bushing H, and in this manner may be placed at an arbitrary azimuth; for any individual

*) The image is taken from the "Katalog mathem. Modelle, Apparate und Instrumente," by commission of the deutschen Mathem.-Vereinig., edited by W. Dyck, appendix page 79.

experiment, however, it is fixed, since friction in the bearing hinders any self-activated rotation of the bracket.

The apparatus is adjusted so that the axis AA' stands exactly horizontal at an arbitrary azimuth with respect to the meridian. The center of gravity of the rotor and frame is first brought, by the adjustment of the screws vv' and the added masses uu', to the line that connects the

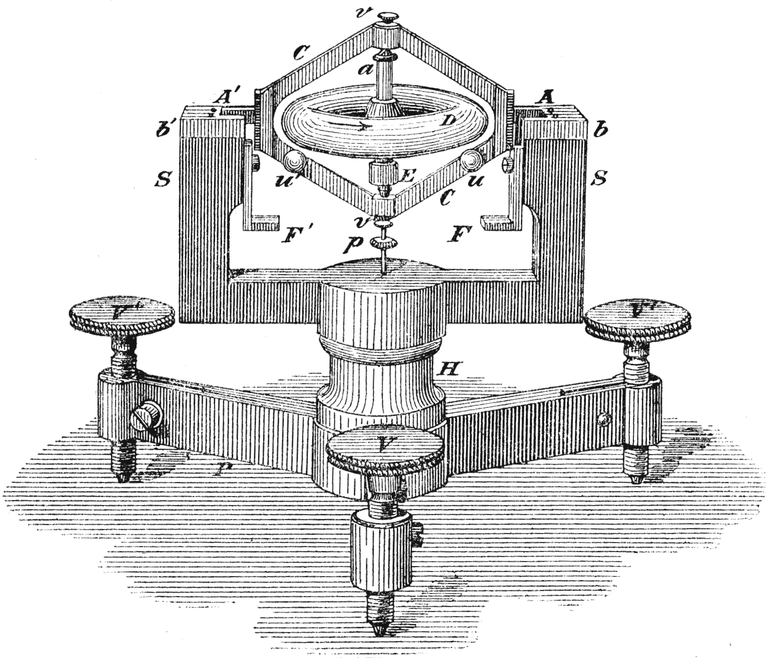

Fig. 111.

knife edges AA', so that the mobile part of the apparatus is in neutral equilibrium. The sliding weight p is then brought to the lower end of the needle that is fixed to the frame, so that the neutral equilibrium is changed to a (slightly) stable equilibrium.

After the rotor has been given a large eigenimpulse N (for G i l b e r t, 150 rotations per second), one observes that its axis moves in an oscillatory manner away from the vertical, and remains, after the oscillation has expired, at a certain inclination that depends, among other things, on the initially positioned azimuth of the frame.

The calculation of this inclination and theory of the experiment are again extremely simple if one begins from the concept of the deviation resistance.

We can speak qualitatively in the following manner. The rotor that moves in an arbitrary vertical plane is under completely similar conditions to those of the rotor in the second Foucault experiment (see above, under b) or c)); in order to fix its position within the vertical plane or its position with respect to the rotating Earth, it would be necessary to overcome the moment K, which has for its axis the connecting line AA' of the knife edges. Since a counter moment $-K$ is not exerted, the rotor moves as if a moment K acted on it about the named axis. This moment strives to place the figure axis of the rotor parallel to the rotation axis of the Earth; it therefore deflects the initially vertical axis of the rotor toward the direction of the projection of the Earth's axis of rotation onto the plane of motion of the rotor axis. In addition to this moment, however, the moment of gravity now acts, which deflects the figure axis of the rotor back toward the vertical. There will thus be a certain mean position between the vertical and the projection of the Earth's axis in which the two turning moments maintain equilibrium, and in which the axis of the rotor is thus at rest.

In order to complete the deliberation from the quantitative side, it is necessary to express the moment of gravity M, on the one hand, and the inertial effect K, on the other hand, in terms of quantities that can be observed in the apparatus.

Let m be the mass of the sliding weight p, and δ its distance from the center of gravity of the rotor. If χ signifies the angle that the figure axis of the rotor forms with the vertical, reckoned positive from the vertical toward the figure axis, then the moment arm of the gravitational force mg about the axis of the knife edges will be $\delta \sin\chi$, and thus the moment of gravity in the sense of the angle χ is

$$M = -mg\delta \sin\chi. \tag{10}$$

For the determination of the deviation resistance K, we again begin from equation (1) of page 175, introduce the eigenimpulse N of the rotor, and set for ν the component $\nu = \omega\cos\lambda$ of the Earth's rotation with respect to the plane of motion of the rotor axis, where λ signifies, as in c), the angle between the Earth's axis and this plane. We thus obtain, if we again neglect a term of the relative order of magnitude $A\omega/N$,

$$K = -N\omega\cos\lambda \sin\vartheta.$$

Here ϑ is the angle between the projection of the Earth's axis onto the plane of motion of the rotor axis and the rotor axis, reckoned as positive

from the former to the latter. The moment K is reckoned in the same sense as ϑ. If we introduce the angle μ that the projection of the Earth's axis onto the plane of motion forms with the vertical, likewise reckoned as positive from the former to the latter, then we have $\vartheta = \mu + \chi$, and can write

$$K = -N\omega(\cos\chi\cos\lambda\sin\mu + \sin\chi\cos\lambda\cos\mu). \tag{11}$$

Here the products $\cos\lambda\sin\mu$ and $\cos\lambda\cos\mu$ are to be expressed in terms of quantities that permit of more direct measurement than the angles λ and μ. We choose as such quantities the geographic latitude φ of the observation location and the angle α ($< 180°$) that the plane of motion of the figure axis forms with the meridian of the observation location. On the unit sphere described about the midpoint of the rotor, we mark (cf. Fig. 112) the intersection point V with the vertical, the intersection point P with the parallel to the axis of the Earth, and finally the point Q that corresponds to the projection of the Earth's axis onto the plane of motion of the rotor axis. The corresponding spherical triangle PQV has a right angle at Q. Its hypotenuse is $\pi/2 - \varphi$, and its sides are λ and μ. The angle at V is equal to α.[270] According to Neper's rule, the two equations

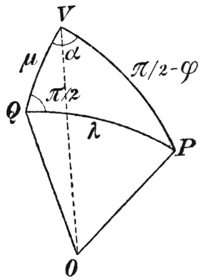

Fig. 112.

$$\begin{aligned} \sin\varphi &= \cos\lambda\cos\mu, \\ \cos\alpha &= \operatorname{tg}\varphi\ \operatorname{tg}\mu \end{aligned} \tag{12}$$

hold, from which follows, by multiplication,

$$\cos\alpha\cos\varphi = \cos\lambda\sin\mu. \tag{12'}$$

If we insert the values of $\cos\lambda\cos\mu$ and $\cos\lambda\sin\mu$ determined in (12) and (12′) into (11), there follows

$$K = -N\omega(\sin\chi\sin\varphi + \cos\chi\cos\varphi\cos\alpha). \tag{13}$$

We can now determine the equilibrium position, as well as the law of motion of the axis of the rotor, in the simplest manner.

The axis of the rotor is in *equilibrium* if the two moments M and K cancel each other. The equation $M = K$ therefore serves for the determination of the equilibrium position. If we denote the value of χ that corresponds to the equilibrium position by χ_0, then we have for the determination of χ_0, according to (10) and (13), the equation

$$mg\delta\sin\chi_0 = N\omega(\sin\chi_0\sin\varphi + \cos\chi_0\cos\varphi\cos\alpha),$$

or

$$\operatorname{tg}\chi_0 = \frac{N\omega\cos\varphi}{mg\delta - N\omega\sin\varphi}\cos\alpha. \tag{14}$$

More generally, we find the *acceleration equation* of the rotor axis if we set the product of the moments of inertia of the moving components and the acceleration of the angle ϑ, or, equivalently, the acceleration of angle χ, equal to the difference of the applied moments. If A is the equatorial moment of inertia of the rotor and A_1 is the moment of inertia of the frame about the connecting line of the knife edges, then the relevant moment of inertia for the change of the angle χ is the sum $A + A_1 + m\delta^2$. The equation of motion is thus

$$(A + A_1 + m\delta^2)\chi'' = K - M,$$

or, according to (10) and (13),

$$(A + A_1 + m\delta^2)\chi'' = -(N\omega \sin\varphi - mg\delta)\sin\chi - N\omega\cos\varphi\cos\alpha\cos\chi.$$

We can write this more conveniently, if we consider the definition of χ_0, as

$$(15)\quad (A+A_1+m\delta^2)\chi''=-\sqrt{(N\omega\sin\varphi-mg\delta)^2+(N\omega\cos\varphi\cos\alpha)^2}\,\sin(\chi-\chi_0).$$

This equation, evidently, can also be identified with the equation of motion of an ordinary pendulum (see above, equation (4′)). The corresponding pendulum length becomes, in analogy to equation (5),

$$l = \frac{g(A + A_1 + m\delta^2)}{\sqrt{(N\omega\sin\varphi - mg\delta)^2 + (N\omega\cos\varphi\cos\alpha)^2}}.$$

The axis of the barogyroscope therefore oscillates about the equilibrium position χ_0 with a period that corresponds to the so-defined pendulum length l. With consideration of the friction at the knife edges, the oscillations will gradually expire, and the final position of rest will coincide with the equilibrium position χ_0.

The formulas (14) and (15) are adopted by Gilbert as the calculational basis for the dimensioning of his apparatus. The Gilbert derivation*) of these formulas is essentially more complicated than that given here. We discuss the results in an obvious manner.

According to equation (14), the inclination χ_0 depends on the azimuth of the assembly, and will be zero, for example, if $\alpha = \pi/2$; that is, if the connecting line of the knife edges AA' lies in the meridian. This follows immediately from the fact that the projection of the Earth's axis onto the plane of motion of the rotor axis coincides with the vertical for this position of the apparatus, and that, therefore, the two moments M and K strive together to position the axis of the rotor vertically. In order, on the other hand, to obtain the most perceptible

*) Cf. the repeatedly cited work Mémoire sur l'application, §XVIII, eqn. (153). The results are summarized in the likewise cited Katalog mathem. Modelle etc., Nachtrag, p. 78

deflection χ_0, the connecting line of the knife edges is directed at a right angle to the meridian, so that the plane of motion of the rotor axis coincides with the meridional plane. In this plane, the needle fixed to the lower end of the rotor will deflect to the south or the north, according to whether this end is, in the manner of expression of page 734, a south or a north pole; that is, according to whether the rotation of the rotor about the needle follows in the opposite or in the same sense as the rotation of the Earth about the north pole of the Earth.

Moreover, equation (14) shows that the amplitude of the deflection depends only on the ratio $mg\delta/N\omega$, and that one would obtain the greatest amplitude (namely, a full right angle) if one were to choose this ratio directly equal to $\sin\varphi$. On the other hand, equation (15) shows that this choice is not the most advantageous in practice (completely disregarding that the support of the frame on the knife edges would make such a great displacement impossible). The length of the corresponding mathematical pendulum would then be, namely,

$$l = \frac{g(A + A_1 + m\delta^2)}{N\omega\cos\varphi\cos\alpha},$$

and the corresponding half oscillation period, for sufficiently small oscillations about the equilibrium position, would be equal to

$$\tau = \pi\sqrt{\frac{A + A_1 + m\delta^2}{N\omega\cos\varphi\cos\alpha}}.$$

For the purpose of a rough calculation, we assume the most favorable case $\cos\alpha = 1$, $\cos\varphi = 1$ (observation in the meridian at the equator) for the magnitude of the amplitude; further, the moment of inertia C of the rotor, which is approximately twice as large as the equatorial moment A of the rotor, may be taken, for example, equal to $A + A_1 + m\delta^2$. If n denotes the number of rotations of the rotor per second, then one has $N = 2\pi Cn$, and

$$l = \frac{g}{2\pi n\omega}, \qquad \tau = \pi\sqrt{\frac{1}{2\pi n\omega}}.$$

For Gilbert (see above), $n = 150$; the value of ω in seconds is then $2\pi/24 \cdot 60 \cdot 60$. Thus $2\pi n\omega$ will equal approximately $10/144$, and

$$l = 144 \text{ m}, \qquad \tau = 12 \text{ sec}.$$

This oscillation period is undesirably long, since the observation must be restricted to the first minutes after the actuation of the rotor, and since one wishes to obtain a judgment concerning the definitive equilibrium

position of the rotor within this time. The strong oscillation of the axis of the rotor (in the assumed case the amplitude of the oscillation amounts to a right angle) would also be inconvenient for the observation of the mean equilibrium position. It is to be added that our value of τ is valid only for sufficiently small oscillation amplitudes, and that, in contrast, the oscillation period in our example would be significantly longer. Nevertheless, our calculation shows that one can arbitrarily enhance the deflection from the vertical through an appropriate choice of the circumstances, and can regulate it according to requirements.

Gilbert himself performed calculations for the numerical example $\varphi = 48^\circ\,50'\,39''$, $\alpha = 0^\circ$, $n = 200$, $m = 0{,}79$ gr., $\delta = 5$ cm.; with consideration of the dimensions of the rotor and the frame, there followed

$$\chi_0 = 7^\circ\,37'\,10'', \quad \tau = 3{,}76 \text{ sec.}$$

Here, therefore, a very large value of the deflection has been renounced in favor of a diminishment of the oscillation period, and the resulting more convenient observation of the final equilibrium position.

The Gilbert disposition appears to have many advantages compared to the original disposition of Foucault. In the sliding weight p and its distance δ from the center of gravity, one has, to a certain extent, a disposable parameter that can be chosen in a favorable manner for the observation. If one adjusts the proportions so that the final equilibrium position lies near the vertical, one eliminates many observational errors that would appear for large amplitudes. Further, the unavoidable errors in the centering of the moving masses, which for the Foucault gyroscope may be very disturbing, become relatively insignificant for the barogyroscope by means of the intentional addition of the overweight p. Concerning the quantitative coincidence between theory and experiment, however, G i l b e r t gives, just like F o u c a u l t, no more detailed information in the repeatedly cited treatise.[271]

In the preceding presentation, we have intentionally emphasized the probable error sources and the question of a possible quantitative confirmation more strongly than is otherwise common in textbooks of mechanics. Indeed, the present example appears eminently suited to showing how long is the path that extends between the mental conception of a dynamic process and its realization by a specific physical apparatus!

The genius of Foucault predicted, more through intuition than strict mechanical deduction, the behavior of the spinning rotor on the surface of the Earth. In order to complete his gyroscope, eight months of intense work was necessary. In spite of the applied labor and the unusual experimental capability of Foucault, the apparatus could operate only on the boundary at which the mechanical truth to be proven begins to be elevated above the disturbances and observational errors. How mistrustful Foucault himself believed one must be with respect to his apparatus emerges from a remark in the above cited "Instructions": One may not be fully convinced that the observed deflection of the gyroscope is actually due to the rotation of the Earth until one has not obtained the same sense of the deflection for the opposite rotation sense of the rotor. It thus appears that the sense of the deflection was never raised above all doubt; that the magnitude of the deflection would be given correctly with a certain precision appears all the more doubtful, especially if the experiment were made by an experimenter less skillful than Foucault.

The chances are more favorable, on the above-named grounds (the lack of the necessity for a particularly precise centering, the selection of more convenient experimental conditions through the appropriate specifications of the overweight), for the barogyroscope. Here too, however, an actual experimental proof that the sources of error do not excessively distort the observed deflection is required, a proof that we seek to no avail in Gilbert.

For a repetition of the Foucault or the Gilbert experiment, one would well, in any case, introduce electromagnetic actuation of the rotor, and thus eliminate the difficulties that arise from the gradual slowing of the eigenrotation.[272]

Addenda and Supplements.[273]

To Chap. VII.

To p. 537. For further information on the history of the laws of friction, especially concerning the older literature, cf. P. Stäeckel, Encykl. d. math. Wiss. IV, 6, Nr. 6.

To Chap. VIII.

To §9. Foucault's idea of the demonstration and measurement of the Earth's rotation by top experiments, which has found its most complete realization in the Anschütz gyrocompass, has also been pursued to quantitative measurements by Föppl (Sitzungsber. d. Akad. d. Wiss. München, Bd. XXXIV, 1904, p. 5, and Phys. Ztchr. 5, 1904). The original Foucault experiments and the later experiments of Gilbert remained far removed from this goal, and the former never once gave a qualitatively certain result. The disposition of Föppl's experiment was closely related to that of Anschütz; the horizontal equilibrium of the nonrotating top, which Anschütz achieved by a counterweight, was accomplished here by a suspension from three wires. The top experienced, just as the gyrocompass did, a directional moment $N\omega \cos\varphi \sin\psi$ that turned it away from its original equilibrium position toward the north (cf. p. 857), while it tended to return to this position because of the trifilar moment and the torsional elasticity of the suspension wires. An initial east–west position ($\psi = 90°$) is most favorable here; the top subsequently takes a position measurably different from $\psi = 90°$, in which the directional moment maintains equilibrium with the moment about the vertical that is generated by the suspension. For a known impulse and a measured directional moment, the rotational velocity ω of the Earth can be determined. As in the Anschütz gyrocompass, it is essential that three degrees of freedom (provided here by the flexibility of the suspension wires) be present, and that the oscillation period be elongated by the "effective inertia" (cf. p. 857). On the other hand, the electromagnetic actuation that makes possible a sufficiently long observation time is naturally also of great significance. The experiment was arranged so that it gave an oscillation amplitude of 5° to 8°, with a period of 6 to 8 minutes.[274]

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Translators' Notes.

200. (page 532) The Model 88 infantry rifle was used by the German army from 1892 to 1915. Sommerfeld likely used this rifle during his military service with the Königsberg reserve regiment in 1892–1893.

201. (page 537) The experiments of Charles-Augustin de Coulomb (1736–1806) on sliding and static friction are described in the first part of his *Théorie des machines simples* [Coulomb 1821]. Coulomb's experimental apparatus is shown in Fig. 183. A "very solid table" supports a fixed oak plank $aa'bb'$ that was "dressed with great care by a plane, and then polished with a shark skin." The plank is 8 *pieds* (2.6 m) long

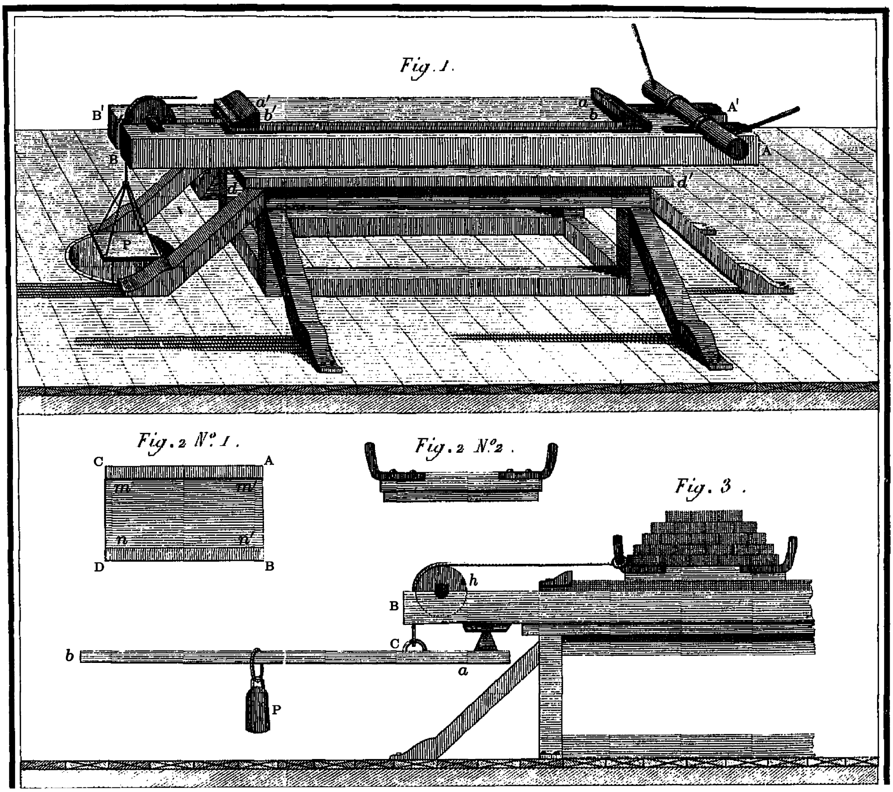

Fig. 183. Coulomb's experimental apparatus for the investigation of sliding and static friction [Coulomb 1821, plate I].

F. Klein and A. Sommerfeld, *The Theory of the Top Volume III: Perturbations. Astronomical and Geophysical Applications*, DOI 10.1007/978-0-8176-4828-2,

and 3 *pouces* (81 mm) thick; its edge is ruled in divisions of one *pouce*. The sled (*traîneau*) that slides on the oak plank has wooden rails on its lower surface, and can be loaded with up to 2 474 *livres* (12 000 N). The sled is pulled by a rope that is attached to the loaded plate P in the sliding friction experiments, and to the balance *acb* in the static friction experiments.

To investigate sliding friction, Coulomb first loads the sled, and then slowly increases the weight on the plate P until a small hammer blow will set the sled in motion. To measure the motion of the sled, an assistant counts aloud the oscillations of a pendulum clock that beats half-seconds, while a second assistant announces by a shout the passage of the sled through each division on the plank. Coulomb himself records the correspondence of the space and time measurements. The velocity of the sled in Coulomb's experiments ranges from 1/7 *pieds*/sec (0.05 m/sec) to 4 *pieds*/sec (1.3 m/sec). Coulomb states that the sled "almost always had a velocity greater than that of all machines that are in use."

Coulomb finds that the time required for the sled to travel the first two *pieds* is always about twice the time required to travel the second two *pieds*. (The sliding distance is generally limited to 4 *pieds* in order to avoid velocities and collisions that would require "great precautions for the safety of the observers.") Since the two travel times would have the ratio $(1+\sqrt{2}):1$ (Coulomb gives this ratio as $100:42$) in a uniformly accelerated motion, Coulomb concludes that the sled has an approximately constant acceleration, and that the friction force must therefore be constant during the motion. From the measured acceleration and the known weights of the sled, plate, and pulley, Coulomb calculates the force due to sliding friction. By varying the weight on the sled and the area of its rails, he further concludes that the friction force depends primarily on the total normal force between the sled and the plank; the friction force is very nearly independent of the pressure over a range of 188 to 1 788 *livres* per square *pied*.

The investigation of static friction is simpler. The weight P on the balance $b\mathrm{C}a$ is adjusted until the loaded sled just begins to move. The known weights of P and the balance arm can be used to determine the tension in the rope, and the ratio of the sled weight to the rope tension can then be computed. Coulomb observes that the static friction force depends on the time that the sled and plank are allowed to remain in contact at rest before the motion is initiated; the static friction force increases with the time of repose for the first few minutes, and then ceases to change.

202. (page 537) Perry's *Applied Mechanics* [Perry 1907, subtitled "A Treatise for the use of Students who have time to work Experimental, Numerical and Graphical Exercises illustrating the subject"] is full of personal comments on the teaching of mechanics, one of Sommerfeld's occupations in Aachen at the time he was writing Vol. III of the *Theorie des Kreisels.* A few of Perry's comments:

> When we think of what goes on under the name of **teaching** we can almost forgive a man who uses a method of his own, however unscientific it may be. Nevertheless, it is not easy to forgive men who, because they have found a study interesting themselves, make their students waste a term upon it, when only a few exercises are wanted—on what is sometimes called the scientific study of arithmetic, for example, or of mensuration.
>
> In our own subject of Applied Mechanics there are teachers who spend most of the time on **graphical statics**, or the graphing of functions on squared paper, or the cursory examinations of thousands of models of mechanical contrivances. One teacher seems to think that applied mechanics is simply the study of **kinematics** and mechanisms; another, that it is simply exercise work on pure mechanics; another, that it is the breaking of specimens on **a large testing machine**; another, that it is the trying to do in a school or college what can only be done in real engineering works; another, that it is mere graphics; another, that it is all calculus and no graphics; another, that it is all shading and colouring and the production of pretty pictures without centre lines or dimensions. Probably the greatest mistake is that of wasting time in a school in giving the information that one cannot help picking up in one's ordinary practical work after leaving school.
>
> We believe that the principles which an engineer really recollects and keeps ready for mental use are very few. By means of lectures, models, drawing-office and laboratory, and numerical exercise work, we show a man how these simple principles enter, in curiously different-looking shapes, into his engineering practice. We give him the use of all the necessary methods of study, and we send him out into practical life prepared to study things for himself. We ought to recognize that his real study of his profession is not at

school or college. We ought to teach him how to learn for himself. Any child can state Newton's second law of motion, and the other half-dozen all-important principles of mechanics, so as to get full marks in an examination paper; the engineer knows that the phenomena he deals with are exceedingly complex, and that only a long experience will enable him to utilise the so easily stated principles. Schools and colleges are the places in which men ought to learn the uses for all mental tools; they are sure to specialise afterwards, but in the meantime we ought to give them plenty of tools to choose from. The average student cannot take in more than the elementary principles, the best students need not take in more [pp. 1–2].

We have said that the **mechanical principles** which must be studied by the young engineer are **few in number**, but they must be very familiar to him. It is not well to say that one method of study is more important than another, the fact being that a student must not only study in the workshops and drawing-office, but he must read, work numerical exercises, and **make a great many quantitative laboratory experiments** to illustrate these principles. Our aim is to get students to think, and it is astonishing how difficult it is to effect this object. We cannot easily get students to wrangle over these subjects. We have few pretty lecture experiments. Even in chemistry and experimental physics, pretty lecture experiments are not very effective in causing students to think. Students will think about things that they do. Hence it is that boys should be allowed to **chip and file metals**, and to pare and cut wood. Merely in learning how to hold a chipping-chisel or in setting a plane iron, a student must think about the properties of materials and forces. Country boys who make their own things have a great advantage over town boys who buy their things in shops. In the mechanical laboratory, I find that even the dullest student begins to think for himself if he is not too much spoon-fed; and if his difficulties are not cleared away by some wretched routine system of laboratory work being adopted by cheap laboratory instructors, the fundamental principles of mechanics will become part of his mental machinery [p. 56].

Perry writes that his own method of teaching, which he practiced from 1875 to 1878 in the Imperial College of Engineering in Tokyo, was "met with some ridicule" when he returned to England.

203. (page 537) Francesco Masi (1852–1944) was professor of applied mechanics in the University of Bologna. Most of his book *Le nouve vedute nelle ricerche teoriche ed esperimentali sull' attrito* is devoted to the hydrodynamic lubrication theory of Nikolai Pavlovich Petroff (1836–1920), a professor in the St. Petersburg Technical Institute. Masi attributes the first investigation of friction between solid bodies to Leonardo da Vinci (1452–1519) [Masi 1897, p. 5].

204. (page 538) Gustav Friedrich Herrmann (1836–1907) was professor of mechanics in the *Technische Hochschule* in Aachen, and was rector there from 1889 to 1892. Herrmann's beautiful little book *Der Reibungswinkel* [Herrmann 1882] was a presentation from Aachen to the University of Würzburg on the occasion of its 300$^{\text{th}}$ anniversary (the university was actually founded in 1402, but was closed in about 1420 due to economic and political difficulties [Wegele 1882]). Herrmann's illustration of the sliding stick experment, which he describes as "well known," in shown in Fig. 184. The experiment can be made more striking by weighting one end of the stick, so that the center of gravity is displaced from the geometric center.

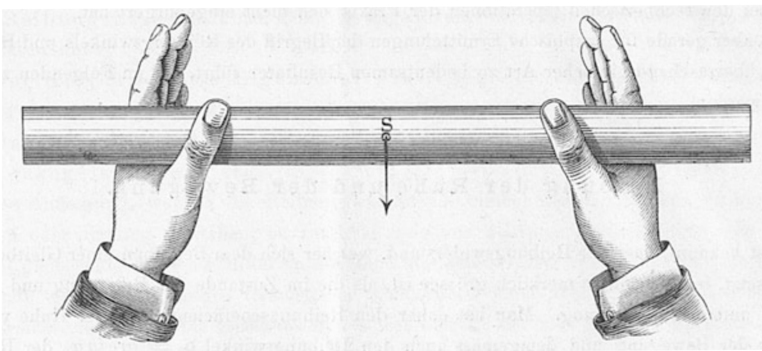

Fig. 184. Sliding stick experiment to illustrate the difference between static and dynamic friction [Herrmann 1882, p. 4].

205. (page 539) Henry Charles Fleeming Jenkin (1833–1885) was professor of engineering in the University of Edinburgh. Sir James Alfred Ewing (1855–1935) was a student of Jenkin's who taught in Tokyo, Dundee, and Cambridge. He was part of the successful effort to decrypt German naval messages in the First World War, and became vice-chancellor of Edinburgh University in 1916.

The experimental apparatus of Jenkin and Ewing is shown in Fig. 185. A cast iron disk A with radius one foot, thickness 3/4 inch, and weight $W = 86$ pounds is supported by very thin spindles (radius $r = 0.05$ inch) in rectangular bearings. When the disk is spun gently in the clockwise direction, it rolls until it reaches the right-hand side of the bearing. The spindles then slide against the horizontal and vertical surfaces of the bearing until the disk comes to rest. The friction to be measured is the friction between the steel spindle and the various materials of the bearings. Because the radius of the spindle is small, the sliding velocity at the frictional interfaces is also small; the velocity range considered by Jenkin and Ewing is 0.01 to 0.0002 feet per second.

Assuming that the center of mass of the disk does not move as the spindle slides in the bearing, the normal forces P_v and P_h on the spindle are

$$P_v = \frac{W}{1+\mu^2}, \quad P_h = \frac{\mu W}{1+\mu^2},$$

where μ is the coefficient of sliding friction; the frictional moment M due to the sliding on the bearing is thus

$$M = \mu r W \Big(\frac{1}{1+\mu^2} + \frac{\mu}{1+\mu^2} \Big),$$

or

$$\mu^2 + \mu \frac{Wr}{Wr - M} = \frac{Wr}{Wr - M} - 1. \tag{1}$$

The coefficient of friction μ can thus be obtained by solving a quadratic equation (which has only one positive root) if the weight W, the spindle radius r, and the frictional moment M are known.

Jenkin and Ewing state that moments on the disk due to air resistance and the contact with the end plates c can be safely neglected compared with the frictional moment M. In this case, the frictional moment M can be calculated from the angular acceleration of the disk as it slowly spins to a halt. The angular acceleration is measured by the electrostatically driven ink spray system (an apparent anticipation of our ink jet printers!) that is attached to the pendulum arm C. As

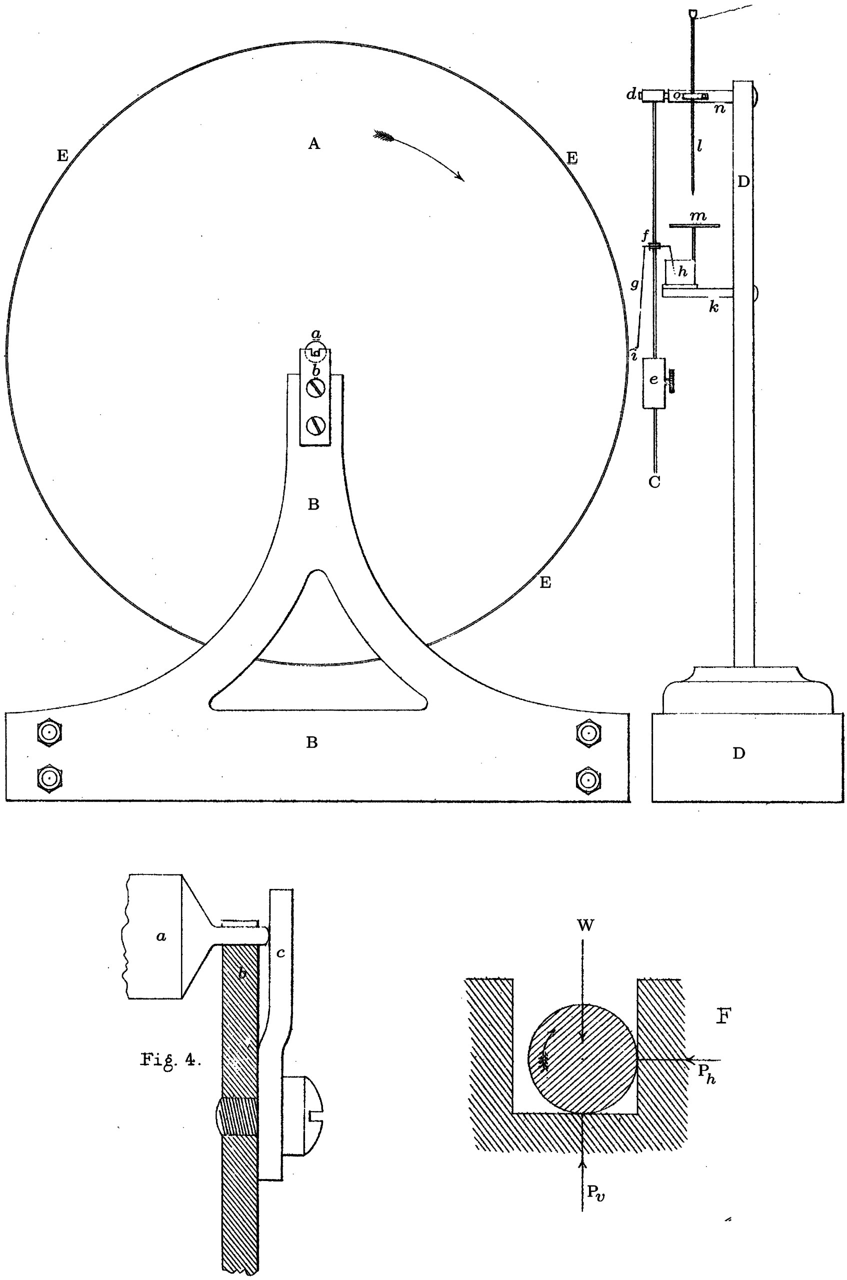

Fig. 185. Jenkin and Ewing's experimental apparatus for the investigation of low-speed sliding friction [Jenkin 1877, plate 20].

the pendulum swings with a known period in the plane perpendicular to the drawing, the ink jet system sprays a sinusoidal curve on a paper strip E that is attached to the edge of the disk. The curve is used to calculate the angular acceleration, and the friction coefficient is then calculated from equation (1).

Jenkin and Ewing cite a paper by the English engineer George Rennie (1791–1866) that summarizes many early experiments on sliding friction [Rennie 1829].

206. (page 540) General Arthur Jules Morin (1795–1880) studied engineering at Metz with Jean Victor Poncelet (1788–1867), and later became director of the Conservatoire National des Arts et Métiers in Paris. Morin's experiments on friction are described in his two-volume memoir *Nouvelles expériences sur le frottement* [Morin 1832, 1834].

207. (page 541) The instrumented van of Sir Douglas Strutt Galton (1822–1899) is illustrated in Fig. 186. The figure is taken from *The Effect of Brakes upon Railway Trains*, a collection of Galton's work on brake friction that was published by the Westinghouse Company [Galton 1894].

The 1878 meeting of the British Institution of Mechanical Engineers at which Galton read his first paper was held at the Conservatoire National des Arts et Métiers in Paris. General Morin was present, as was George Westinghouse, Jr. (1846–1914), the American inventor of the railcar air brake. The discussion that followed the reading of Galton's paper included a suggestion for the development of an antilock brake system to prevent skidding on the rails, which Galton found to greatly reduce the braking effectiveness. At the conclusion of the discussion,

> The PRESIDENT said he would ask the members to pass a vote of thanks to Captain Galton for his interesting paper. They ought not to lose sight of the fact that, although the paper was short (which in some respects was desirable), it was of great value. It was not the length of a paper which indicated the amount of labor that had been spent upon it; and every member present would understand the amount of time, expense, and patience required to get all these facts placed before them in the way that Captain Galton had done. He asked the members to thank him, not only for what they had then received, but for the good things he had promised in the future.
>
> The vote of thanks was passed by acclamation.

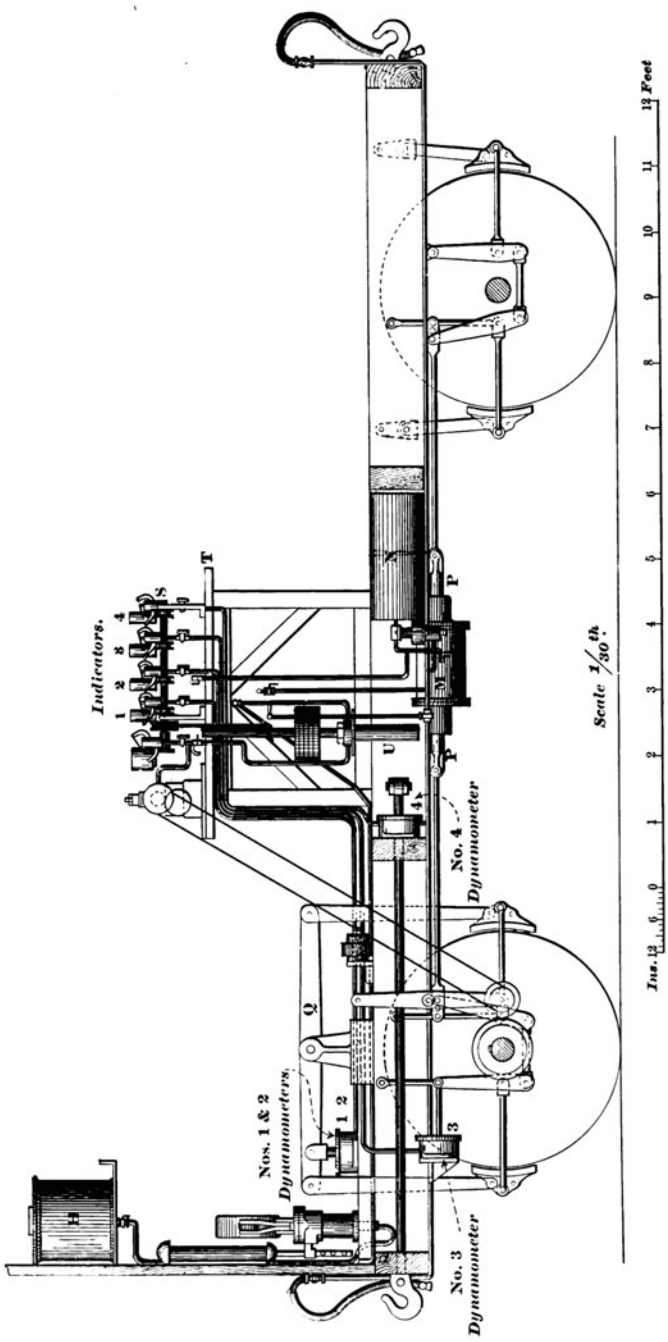

Fig. 186. Galton's experimental van for the determination of brake pad friction [Galton 1894, Pl. 58].

208. (page 542) A discussion of the cited experiments is given in Sommerfeld's paper *Zur Theorie der Eisenbahnbremse* [Sommerfeld 1902c]. One of the goals of Sommerfeld's paper is to explain the observed fact that an increase of the pressure between the brake pad and the wheel can sometimes lead to a decrease in the effectiveness of the braking. This phenomenon is explained by a dynamic analysis in which the wheel may slide on the rail as well as turn on its axle.

209. (page 546) Richard Hermann Stribeck (1861–1950) was professor of machine engineering in Darmstadt and Dresden. In 1898, he became the director of the Center for Scientific and Technical Research, an industrial laboratory in Neubabelsberg near Berlin. Sommerfeld wrote an important paper on hydrodynamic lubrication theory [Sommerfeld 1904] in which he compares his theoretical results for bearing friction to the experiments of Stribeck, which are still quoted in textbooks on tribology today.

210. (page 550) We have corrected Fig. 75; in the original, the arrow that represents the vertically directed moment is labeled M_1, and the arrow that represents the horizontally directed moment is labeled M_2.

211. (page 566) A computational reproduction of Sommerfeld's Fig. 79 on p. 564 is shown in Fig. 187. Fig. 187 was produced with the numerical values $m = 1/3$ (the value of m that Sommerfeld uses to construct Fig. 78 on p. 561) and $\lambda\mu = 1/3$. The serpentine curve in Fig. 187 is a numerical solution of the differential equation (15) on p. 562. As usual, Sommerfeld's ingenious construction in Fig. 79 gives an astonishing approximation to the numerical solution. For smaller values of m and $\lambda\mu$, the oscillation of the serpentine curve becomes denser, but the character of the curve is completely unchanged.

212. (page 578) The mechanical engineer Hans Lorenz (1865–1940) was the predecessor of Ludwig Prandtl as the director of the Institute for Technical Physics in Göttingen. He was later appointed to the chair of mechanics at the *Technische Hochscule* in Danzig. Lorenz's comment about the control valve of a steam engine is made in the context of a discussion of particle motion on a rough horizontal plane. A horizontal force applied to a stationary particle on the horizontal x, y plane must overcome the static friction force due to the weight of the particle before a displacement is produced, but, as Lorenz writes, "if a body such as the control valve of a steam engine is moving in the x-direction,

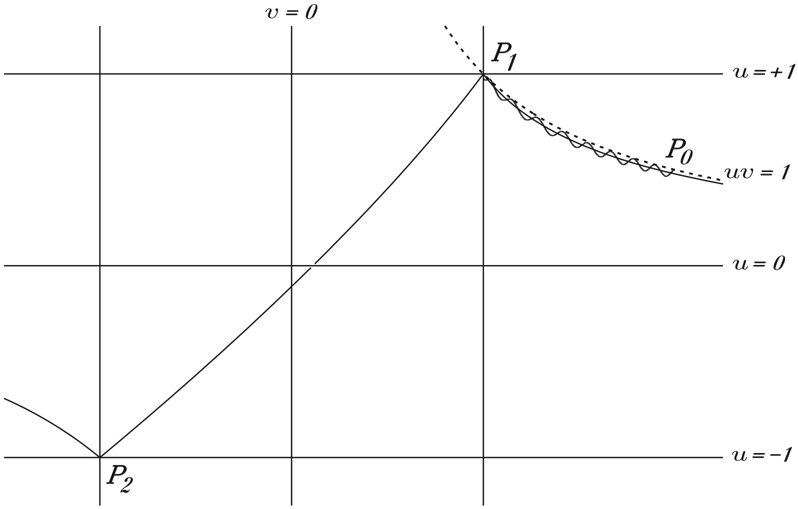

Fig. 187. Numerical solution for the inclination angle of the top with sliding friction at the support point.

then an arbitrarily small force in the y-direction is enough to produce a displacement in the direction perpendicular to the motion, even if an arbitrarily large steam pressure is present instead of the weight mg."

213. (page 581) The once-popular handbook of logarithmic and trigonometric tables compiled by Oskar Xavier Schlömilch (1823–1901), for example, also contains a table that gives the arc length of a quarter-ellipse with unit semi-major axis and semi-minor axis b, for b in the range $0 \leq b \leq 0.99$. This arc length is equal to the complete elliptic integral of the second kind with modulus $k = \sqrt{1-b^2}$ [Schlömilch 1908, pp. 150–151].

214. (page 592) The motion of the figure axis in space and the spatial curve traced by the endpoint of the impulse for the gravity-free symmetric top with air resistance can be obtained from the Lagrange equations for the Euler angles φ, ψ, ϑ. In the notation used by Sommerfeld on pages 593–594, the Lagrange equations for this case are

$$\frac{dN}{dt} = -\frac{\lambda}{C}N, \tag{2}$$

$$\frac{dn}{dt} = -\frac{\lambda}{A}n - \lambda\left(\frac{1}{C} - \frac{1}{A}\right)N\cos\vartheta, \tag{3}$$

$$A\frac{d^2\vartheta}{dt^2}+\lambda\frac{d\vartheta}{dt}=-\frac{(N-n\cos\vartheta)(n-N\cos\vartheta)}{A\sin^3\vartheta}, \tag{4}$$

$$N=C\Big(\frac{d\varphi}{dt}+\cos\vartheta\,\frac{d\psi}{dt}\Big), \tag{5}$$

$$n=A\frac{d\psi}{dt}\sin^2\vartheta+C\cos\vartheta\Big(\frac{d\varphi}{dt}+\cos\vartheta\,\frac{d\psi}{dt}\Big). \tag{6}$$

We assume the initial conditions

$$\varphi(0)=0,\quad \psi(0)=0,\quad \vartheta(0)=\vartheta_0,$$

$$\dot\vartheta(0)=0,\quad N(0)=N_0,\quad n(0)=\frac{N_0}{\cos\vartheta_0},$$

which correspond to a regular precession with the impulse in the vertical (z) direction.

An approximate solution of the Lagrange equations may be obtained by neglecting the time derivatives of ϑ on the left-hand side of equations (4) and assuming that $N=n\cos\vartheta$. This is analogous to the quasistatic approximation discussed by Sommerfeld on p. 594. The solution obtained in this manner is

$$N=N_0e^{-\frac{\lambda t}{C}}, \tag{7}$$

$$n=N_0\,e^{-\frac{\lambda t}{C}}\Big[1+\tan^2\vartheta_0\,e^{-2\lambda t\left(\frac{1}{A}-\frac{1}{C}\right)}\Big]^{1/2}, \tag{8}$$

$$\tan\vartheta=\tan\vartheta_0\,e^{-\lambda t\left(\frac{1}{A}-\frac{1}{C}\right)} \tag{9}$$

$$\varphi=\int_0^t\Big\{\frac{N-n\cos\vartheta}{A\sin^2\vartheta}+\Big(\frac{1}{C}-\frac{1}{A}\Big)N\Big\}\,dt \tag{10}$$

$$=-\frac{2\pi\gamma}{\delta}\Big(1-e^{-\frac{\lambda t}{C}}\Big) \tag{11}$$

$$\psi=\int_0^t\frac{n-N\cos\vartheta}{A\sin^2\vartheta}\,dt \tag{12}$$

$$=\int_0^t\frac{N_0}{A}\,e^{-\frac{\lambda t}{C}}\Big[1+\tan^2\vartheta_0\,e^{-2\lambda t\left(\frac{1}{A}-\frac{1}{C}\right)}\Big]^{1/2}\,dt. \tag{13}$$

According to equation (9), the angle ϑ approaches 0 as $t \to \infty$ for $C > A$ and approaches $\pi/2$ as $t \to \infty$ for $C < A$, in agreement with Sommerfeld's conclusions on p. 592.

The quasistatic solution of the Lagrange equations implies that the impulse remains vertical as its magnitude decreases. In order to obtain an analytic representation for the spiral impulse curve depicted by Sommerfeld in Fig. 88 on p. 591, we set, as Sommerfeld does on p. 574, $\vartheta = \vartheta_1 + \vartheta_2$, where ϑ_1 is the slowly varying quasistatic approximation given by equation (9) and ϑ_2 is a small but rapidly varying correction. There follows from equation (4), if the time derivatives of ϑ_1 are neglected compared to those of ϑ_2 and the right-hand side is expanded to first order in ϑ_2,

$$\frac{d^2\vartheta_2}{dt} + \frac{\lambda}{A}\frac{d\vartheta_2}{dt} + \frac{n^2}{A^2}\vartheta_2 = 0, \tag{14}$$

where n is the quasistatic approximation given in equation (8). Since n varies slowly and the damping is small, the WKB method [Morse 1953, p. 1092] can be used to obtain an analytic approximation for ϑ_2. The first-order WKB approximation is

$$\vartheta_2 = ae^{-\frac{\lambda t}{2A}} \sin\phi, \quad \phi = \int_0^t \frac{n}{A}\, dt. \tag{15}$$

The constant a is chosen so that the sum $\vartheta = \vartheta_1 + \vartheta_2$ satisfies the initial condition $\dot{\vartheta}(0) = 0$. Corrections to the quasistatic approximations for the Euler angles φ and ψ can be obtained by substituting $\vartheta = \vartheta_1 + \vartheta_2$ into equations (10) and (12) and then expanding in the small quantity ϑ_2.

Fig. 188 shows the spiral impulse curve that corresponds to the corrected quasistatic solution for the case $\vartheta_0 = \pi/4$, $\delta = 2\pi\lambda/N_0 = 0.1$, and $\gamma = (C-A)/A = 0.5$. The body frame spiral of the angular velocity vector for this case is shown in Fig. 84 on p. 589. The spiral loops of the impulse curve in Fig. 188 are actually quite small; the scale for the horizontal directions in the figure is 1/50 of the scale for the vertical direction.

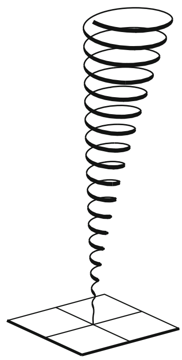

Fig. 188. Spatial impulse curve for the symmetric top with air resistance.

215. (page 592) Edward James Stone (1831–1897) was chief assistant to George Airy at the Greenwich Observatory from 1860 to 1870, and was director of the Royal Observatory at the Cape of Good Hope from 1870 to 1879. One of his major works in South Africa was the completion of a catalogue of 12,000 stars that are visible in the southern hemisphere.

Stone's paper [Stone 1867] on the motion of the Earth's axis is primarily mathematical. He solves the Euler equations for the case in which the frictional moment on the Earth is proportional to its angular velocity (Sommerfeld's case on pp. 586–589), and also for the case in which the frictional moment on the Earth is proportional to the angular velocity of the moon relative to the Earth. Stone makes numerical estimates for displacement of the axis of rotation in both cases, and concludes that "on the whole I am of the opinion that this cause is not available for an explanation of those secular changes of climate which geologists have shown to have taken place on our Earth."

216. (page 598) For the heavy symmetric top with air resistance, the differential equations for the eigenimpulse N, the vertical impulse n, and the inclination angle ϑ are

$$\frac{dN}{dt} = -\frac{\lambda}{C}N, \qquad N = N_0 e^{-\frac{\lambda t}{C}}, \tag{16}$$

$$\frac{dn}{dt} = -\frac{\lambda}{A}n - \lambda\Big(\frac{1}{C} - \frac{1}{A}\Big)N\cos\vartheta, \tag{17}$$

$$A\frac{d^2\vartheta}{dt^2} + \lambda\frac{d\vartheta}{dt} = P\sin\vartheta - \frac{(N - n\cos\vartheta)(n - N\cos\vartheta)}{A\sin^3\vartheta}. \tag{18}$$

If we again make the quasistatic assumption that the time derivatives of ϑ can be neglected, then equation (18) reduces to

$$(N - n\cos\vartheta)(n - N\cos\vartheta) = AP\sin^4\vartheta, \tag{19}$$

or

$$\Big(1 - \frac{n}{N}\cos\vartheta\Big)\Big(\frac{n}{N} - \cos\vartheta\Big) = \frac{AP}{N^2}\sin^4\vartheta = \frac{AP}{N_0^2}e^{\frac{2\lambda t}{C}}\sin^4\vartheta. \tag{20}$$

For initial conditions that correspond to a strong impulse about the figure axis, AP/N_0^2 is small, and $n_0 \approx N_0\cos\vartheta_0$. For these initial conditions, we will derive approximate solutions of equations (17) and (20) for t small ($t \ll C/\lambda$) and t large ($t \gg C/\lambda$).

a) *t small.* When t is small, an approximate solution of (20) for $\cos\vartheta$ is (cf. p. 614)

$$\cos\vartheta = \frac{n}{N} - \frac{AP}{N^2}\Big(1 - \frac{n^2}{N^2}\Big). \tag{21}$$

Substitution of (21) into (17) gives

$$\frac{dn}{dt} = -\frac{\lambda}{C}n + \frac{AP}{N_0^2}e^{\frac{\lambda t}{C}}N_0\lambda\Big(\frac{1}{C} - \frac{1}{A}\Big)\Big(1 - \frac{n^2}{N_0^2}e^{\frac{2\lambda t}{C}}\Big). \tag{22}$$

The solution of (22), accurate to the first order in the small quantity AP/N_0^2, is

$$n = n_0 e^{-\frac{\lambda t}{C}} + \frac{AP}{N_0^2}\frac{N_0}{2}\Big(1 - \frac{C}{A}\Big)\Big(1 - \frac{n_0^2}{N_0^2}\Big)e^{\frac{\lambda t}{C}}. \tag{23}$$

Substitution of (16) and (23) into (21) gives the small time approximation for $\cos\vartheta$ as

$$\begin{aligned}\cos\vartheta &= \frac{n_0}{N_0} - \frac{AP}{N_0^2}\Big(1 - \frac{n_0^2}{N_0^2}\Big)\frac{1}{2}\Big(1 + \frac{C}{A}\Big)e^{\frac{2\lambda t}{C}} \\ &= \cos\vartheta_0 - \frac{AP}{N_0^2}\sin^2\vartheta_0\frac{1}{2}\Big(1 + \frac{C}{A}\Big)e^{\frac{2\lambda t}{C}}.\end{aligned} \tag{24}$$

This result agrees with Sommerfeld's equation (15) on p. 595 for the special case $A = C$. According to (24), $\cos\vartheta$ increases with time (ϑ decreases with time) for $P < 0$, and $\cos\vartheta$ decreases with time (ϑ increases with time) for $P > 0$, independently of the relative magnitudes of C and A.

b) *t large.* When t is large, the impulse components n and N must both approach zero. Equation (19) then implies that $\sin\vartheta$ must approach zero, so that ϑ must approach 0 or π. For $\vartheta \approx 0$, (19) is

$$-(n - N)^2 = AP\sin^4\vartheta, \tag{25}$$

which implies that P must be negative. For $\vartheta \approx \pi$, (19) is

$$(n + N)^2 = AP\sin^4\vartheta, \tag{26}$$

which implies that P must be positive.

For $\vartheta \approx 0$ and P negative, equation (25) gives

$$n - N = \sqrt{-AP}\sin^2\vartheta = \sqrt{-AP}(1 - \cos^2\vartheta),$$

or, since n and N are small,

$$\cos\vartheta = 1 - \frac{n-N}{2\sqrt{-AP}}. \tag{27}$$

Substitution of equation (27) into equation (17) now gives

$$\begin{aligned}
\frac{dn}{dt} &= -\frac{\lambda}{A}n - \lambda\Big(\frac{1}{C} - \frac{1}{A}\Big)N\Big(1 - \frac{n-N}{2\sqrt{-AP}}\Big) \\
&\approx -\frac{\lambda}{A}n - \lambda\Big(\frac{1}{C} - \frac{1}{A}\Big)N \\
&= -\frac{\lambda}{A}n - \lambda\Big(\frac{1}{C} - \frac{1}{A}\Big)N_0 e^{-\frac{\lambda t}{C}},
\end{aligned}$$

from which follows

$$n = k_1 e^{-\frac{\lambda t}{A}} + N_0 e^{-\frac{\lambda t}{C}},$$

where k_1 is a constant that cannot be determined from this large-t analysis. According to equation (27), the approximate formula for $\cos\vartheta$ when t is large is now

$$\cos\vartheta = 1 - \frac{k_1}{2\sqrt{-AP}}e^{-\frac{\lambda t}{A}}, \quad P < 0.$$

When $\vartheta \approx \pi$ and P is positive, equation (26) gives

$$n + N = \sqrt{AP}\sin^2\vartheta = \sqrt{AP}(1 - \cos^2\vartheta),$$

so that

$$\cos\vartheta = -1 + \frac{n+N}{2\sqrt{AP}}. \tag{28}$$

Substitution of equation (28) into equation (17) gives a differential equation that can be solved for n, and the substitution of this solution into (28) then gives the formula for $\cos\vartheta$ when t is large in the form

$$\cos\vartheta = -1 + \frac{k_2}{2\sqrt{AP}}e^{-\frac{\lambda t}{A}}, \quad P > 0,$$

where k_2 is another constant.

Combining our quasistatic results for t small and t large, we conclude that Sommerfeld's Fig. 89 on page 596 is valid for both the oblate ($C > A$) and prolate ($C < A$) heavy top with air resistance.

217. (page 602) We have corrected Fig. 90; in the original, the line through O that forms the angle δ with respect to the horizontal is missing and the angle Θ' is labeled incorrectly.

218. (page 623) Carl Barus (1856–1935) was professor of physics in Brown University from 1895 to 1926. A touching account of his life is given in an obituary by Robert Linsday [Lindsay 1937]. Barus's top is illustrated in Fig. 189. The thin plate a is a tin disk or web that carries a circular ring b of 1/8-inch copper wire. The brass tube d holds the pencil e. The top is free to revolve about the handle f, so that it may be spun by a string wrapped around the tube d. The diameter of the disk a is about 6 inches, and the total weight of the top is about 5 ounces.

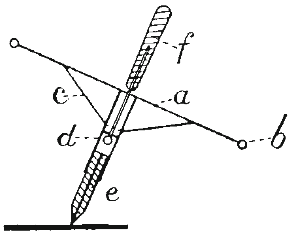

Fig. 189. Barus's curve-tracing top [Barus 1896, p. 444].

One of Barus's support point traces, which he describes as "cornucopia-like," is shown in Fig. 190. The dip indication corresponds to the slope of the support plane, which consists of plate glass, at least a foot square, framed and provided with leveling screws, and covered by smooth white paper. Barus writes that "the tracery at proper angles of dip is exceedingly delicate . . . unfortunately the figure does not convey the finish of the original."

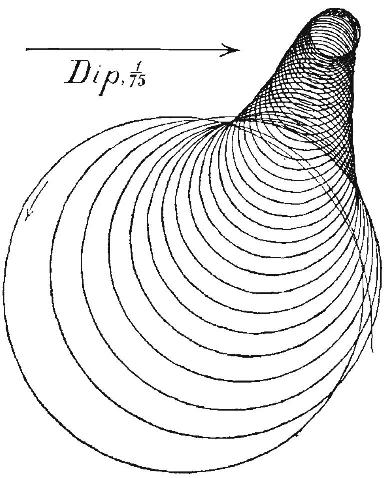

Fig. 190. Support point trace for Barus's curve-tracing top [Barus 1896, p. 445].

Barus also made experiments in which the plane support surface was replaced by a conical surface, which may be either raised or depressed at the center. In this case, "the progressive motion of the top becomes orbital about the axis of the cone, if the dip be suitably chosen."

219. (page 626) The position of the top in Fig. 93 is illustrated more explicitly in Fig. 191. The contact point P between the top and the horizontal plane has the velocity components v and V. The velocity component v due to the precessional motion of the top is tangent to the circle of radius r_0. The velocity component V due to the spin of the top is perpendicular to both the vertical and the figure axis of the top, and is tangent to the contact circle through P. The friction force W is opposite to the vector resultant of v and V, and is parallel to the radius r_s that connects the center of the concentric circles to the vertical projection of the center of gravity S.

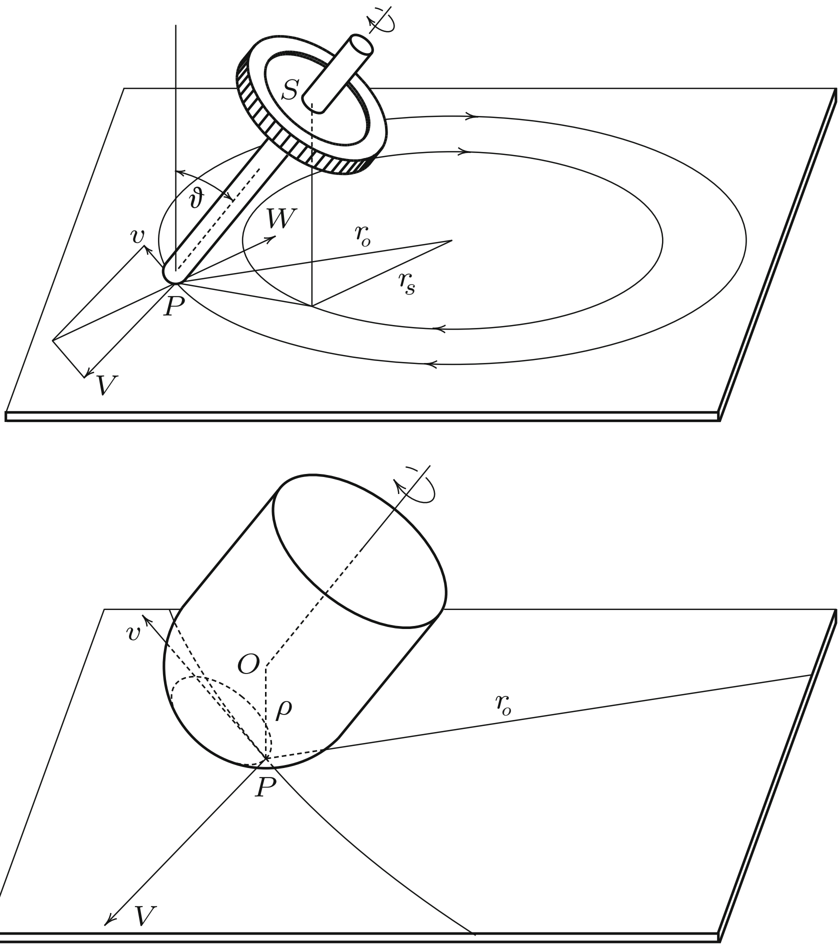

Fig. 191. Top on the horizontal plane
with friction at the support point.

220. (page 631) Archibald Smith (1813–1872) was a fellow of Trinity College, Cambridge, a practicing lawyer, and one the founders of the *Cambridge Mathematical Journal.* In the two-page paper cited by Sommerfeld [Smith 1846], Smith gives a qualitative explanation for the uprighting of the top with sliding friction at the support point. Smith also quotes a previous explanation due to Euler:

> The friction will perpetually retard the motion of the apex of the instrument, and at last reduce it to rest. If this happen before the top fall, it must then be spinning in such a position that the point can remain stationary; but this cannot be if it be inclined. Hence it must have a tendency to erect itself into a vertical direction.

"This reasoning," writes Smith, "is not only of the most vague and inconclusive kind, but is remarkable as being directly the reverse of the truth." Leaving aside the question of how reasoning can be simultaneously vague, inconclusive, and directly the reverse of the truth, Smith's quotation is a brutal abridgment of Euler's argument in the *Theoria motus corporum solidorum seu rigidorum*, *Supplementum de motu corporum rigidorum a frictione perturbato*, Chapter 4, Scholion 2 [Euler 1765, p. 488]:*

> Indeed, this great perturbation of the motion continues until friction ceases. It is evident, however, that this cessation must occur of itself, since friction continually retards the motion. But friction cannot cease unless the cusp of the top should remain in the same place, from which it follows that the motion must be moderated so that the cusp of the top should remain at the same point of the plane, as long as this should occur before the top falls. For if the initial spinning motion impressed upon the top is too slow, there is no doubt that the top should fall before this phenomenon takes place; from which we may conclude, in turn, that if the initial motion will have been sufficiently swift, the cusp of the top is led by friction to the same point of the horizontal plane before the top should fall. When this has occurred and the motion of the top is still rotational, it is clear from the above that the axis of the top ought to be vertical; if it were inclined, the top would in no way be

*We are very grateful to Prof. Patricia Larash of Boston University for her translation of the original Latin text.

> able to spin so that the cusp remains in the same place. From these propositions we deduce this conclusion: if the impressed rotational motion of the top is sufficiently swift, the top finally uprights itself into a vertical position because of friction, and the rotational motion then continues about the vertical axis. This phenomenon is all the more noteworthy since it is due to friction alone. In this way a vertical line, and therefore even a horizontal plane, can be obtained by the work of friction, which can have great use in navigation, as was once noted in England.

Euler's comment on the navigational use of the top probably refers to the horizontal speculum, an instrument invented by the English seaman John Serson. Serson's device is shown in Fig. 192. It consists of a polished circular disk a that spins about the axis b. The diameter of the disk is about three inches. The rounded lower end m of the axis rests in the concave agate cup p. The disk is spun by temporarily attaching the mechanism *lofro* and pulling smartly on the ribbon g. After the mechanism *lofro* is removed, friction at the support point m causes the axis b to become vertical, and thus the disk a to become horizontal. A quadrant Q can then be used to sight both the Sun S and the image s of the Sun in the disk. The angle between the sight line to the Sun and the sight line to the image is twice the angle of the Sun above the (possibly invisible) horizon.

Serson's instrument was tested in September 1743 aboard a royal yacht near the Nore at the mouth of the River Thames. The investigative party on the yacht consisted of Serson, two British naval captains, and Thomas Simpson (1710–1761), master of mathematics at the Royal Military Academy in Woolrich and eponym of Simpson's rule for numerical integration. The results of the tests and the subsequent events are best described in the words of a contemporary source [Bevis 1754]:

> The next day Mr. *Serson* himself undertook to draw up an account of these tryals, but being an illiterate man, did it so improperly that the company refused to sign it. However, the aforesaid captains reported to the commissioners, that in their opinion, Mr. *Serson's* contrivance was highly deserving their encouragement, as likely to prove very useful in foggy weather. Mr. *Serson* having afterwards thought of a method for securing his speculum from tarnish, and the force of the wind, was at length ordered on board his maj-

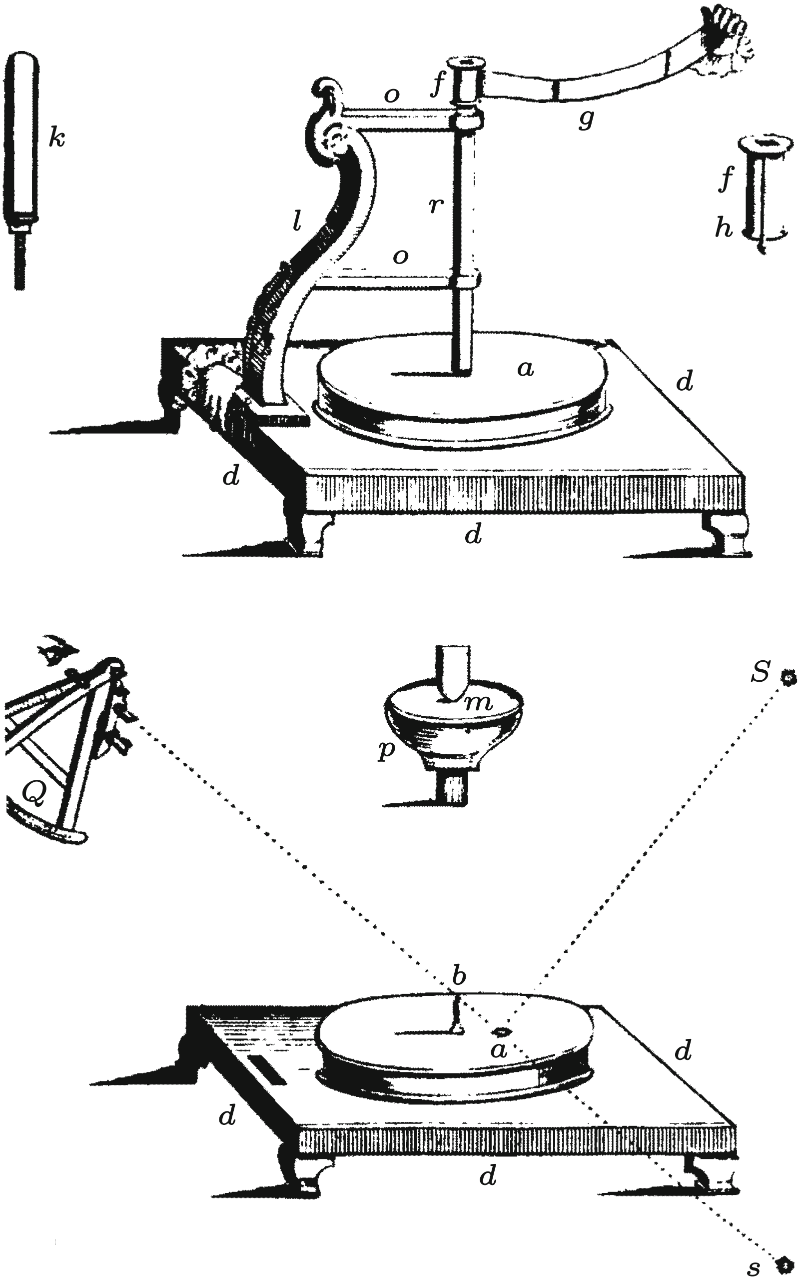

Fig. 192. Sersons' horizontal speculum for determining the altitude of the Sun above an invisible horizon [Bevis 1754].

esty's ship the *Victory*, a first rate, to make observations with his instrument, which were to be compared with those taken, in the usual way, by the ship's officers. But the *Victory* was soon unfortunately lost [sank in the English Channel in 1744], with all on board; and so perish'd poor Mr. Serson, and in some sort his invention too; for I cannot learn that it has been at all prosecuted since, nor do I know that so much as a print of it has hitherto been published.

The English optician James Short (1710–1768) published a paper in the *Philosophical Transactions of the Royal Society of London* on some laboratory experiments with Serson's top [Short 1752]. Short describes the top as "pretty well known," and states that the top spun for thirty-five minutes in the open air and provided a true horizontal plane after about two minutes. The plane "was not at all disturbed by any motion or inclination you give the box, in which it is placed, and therefore might be proper to be used aboard a ship." Short also made experiments with the top in an evacuated chamber, and found that the top spun in the chamber for two hours and sixteen minutes, and "preserved a perfect horizontality for the space of $1/4$ of an hour."

221. (page 631) John Hewitt Jellett (1817–1888) was a fellow and provost of Trinity College, Dublin, and a priest in the Anglican Church of Ireland. The purpose of Jellett's *Theory of Friction* [Jellett 1872] is to incorporate frictional forces into rational mechanics. According to Jellett, "the theory of friction is as truly a part of Rational Mechanics as the theory of gravitation." In order to achieve his goal, Jellett introduces the rather dubious distinctions between *moving* and *resisting* forces, and between *necessary* and *possible* equilibria. His analysis of the top with friction at the support point, which is included in a chapter entitled "Miscellaneous Problems," does not depend on these generalities. Jellett's model of the top on a horizontal plane is shown in Fig. 193. He assumes, as Sommerfeld does, that the lower end of the top has a spherical form. The contact point between the top and the plane may move, but the spin of the top is assumed to be large enough that the sliding friction force at the contact point is approximately equal to the vertical component of the reaction force multiplied by the linear velocity at the contact point due to the component of angular velocity about the figure axis only. Under this assumption, Jellett writes the linear momentum equations and the Euler equations for the top, and

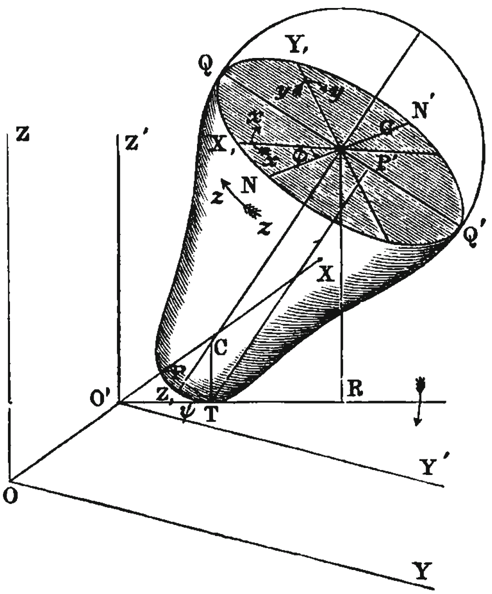

Fig. 193. Jellett's model for the motion of the top on a horizontal plane with sliding friction at the support point [Jellett 1872, p. 181].

is able to show, without completely integrating the equations, that the top becomes upright before any significant fraction of its spin is lost.

222. (page 633) The four-volume *Traité de mécanique céleste* [Tisserand 1889–1896] by François-Félix Tisserand (1845–1896) was considered the definitive treatise on celestial mechanics at the end of the nineteenth century. Tisserand succeeded Victor Alexandre Puiseux (1820–1883) in the chair of celestial mechanics at the Sorbonne. After Tisserand's premature death at the age of fifty-one, the chair was occupied by Henri Poincaré (1854–1912).

223. (page 634) Gauss's treatise *Determinatio attractionis quam in punctum quoduis positionis datae exerceret planeta si eius massa per totam orbitam ratione temporis quo singulae partes describuntur uniformiter esset dispertita* is reprinted in Vol. 3 of his *Werke* [Gauss 1866a, pp. 331–355]. The quotation given by Sommerfeld is taken from the abstract of a lecture that Gauss gave to the Royal Society of Göttingen in January of 1818 [Gauss 1866a, pp. 357–360]. When stating his procedure for calculating the secular perturbation of one planet on another,

Gauss adds the condition that the orbital periods of the two planets should not be rationally related. Gauss writes the gravitational potential of the elliptical ring with a nonuniform mass distribution (see note 226) in terms of an elliptic integral, and presents his method of the arithmetic–geometric mean for evaluating this integral.

224. (page 635) A critical account of Newton's explanation of the precession of the Earth's axis is given by the 1983 Nobel laureate Subrahmanyan Chandrasekhar (1910–1995) [Chandrasekhar 1995, pp. 455–475]. After a detailed discussion of Newton's analysis, Chandrasekhar writes that

> I find it hard to believe that Newton's concluding remark in Proposition XXXIX, that he had accounted for the precession of the equinoxes "agreeing with the phenomena", has invariably—at least, to my knowledge—been accepted with apparent approval, when it should have been obvious, even to a casual reader, that the agreement must be fortuitous: Newton's theory is based on an idealized model of the spheroidal Earth in which the bulges are replaced by an equivalent thin circular disc in the equatorial plane, *albeit* by a treatment of surpassing insight; the choice of $\varepsilon = \frac{1}{230}$ for the ellipticity of the Earth is substantially different from the true value $\frac{1}{289}$ that Newton himself had derived in Proposition XIX (Book III); and finally Newton's choice of $4{\cdot}4815$ for the ratio of the tidal effects of the Moon and of the Sun is at best unreliable (by a factor of ~ 2 as we now know).

225. (page 648) James Bradley (1693–1762) was the third Astronomer Royal of Great Britain, succeeding John Flamsteed (1646–1719) and Edmund Halley (1656–1742). In the process of trying to detect the parallax of the star γ Draconis due to the changing position of the Earth in its orbit around the Sun, Bradley discovered both the nutation of the Earth's axis and the aberration of light due to the Earth's orbital velocity. A description of Bradley's work at the Greenwich Observatory is given by the eleventh Astronomer Royal, Sir Richard van der Riet Woolley (1906–1986) [Woolley 1963].

226. (page 650) To illustrate Gauss's method for computing the gravitational potential of a body that moves in a periodic orbit, we first consider Fig. 194(a), in which a mass m_1 moves in an elliptical orbit around the gravitational focus F. The velocity of the mass is ds/dt,

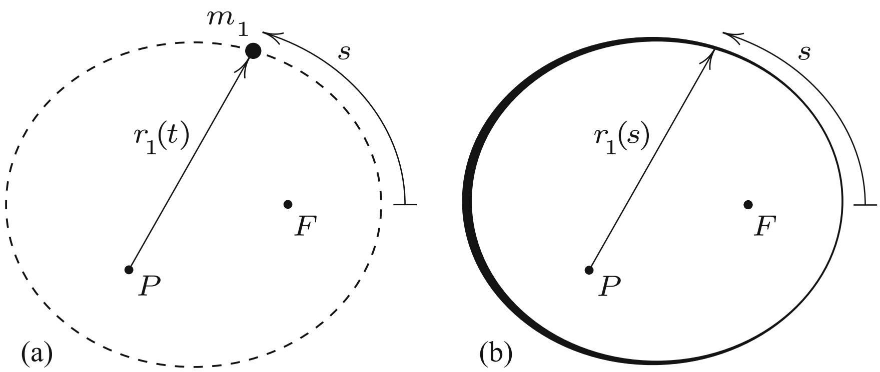

Fig. 194. Elliptical orbit and elliptical ring.

where s is the arc length of the ellipse. The period of the orbit is T_1. The gravitational potential $V_1(t)$ of m_1 at the fixed point P is

$$V_1(t) = -\frac{Gm_1}{r_1(t)},$$

where G is the gravitational constant and $r_1(t)$ is the distance between P and m_1. Since $V_1(t)$ is a periodic function of time with period T_1, it may be expanded in the Fourier series

$$V_1(t) = \overline{V}_1 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{2\pi nt}{T_1} + b_n \sin\frac{2\pi nt}{T_1}\right),$$

where the constant term $\overline{V}_1$ and the coefficients a_n and b_n are

$$\overline{V}_1 = \frac{1}{T_1}\int_0^{T_1} V_1(t)\,dt = -\frac{1}{T_1}\int_0^{T_1}\frac{Gm_1}{r_1(t)}\,dt,$$

$$a_n = \frac{2}{T_1}\int_0^{T_1}\sin\frac{2\pi nt}{T_1}V_1(t)\,dt = -\frac{2}{T_1}\int_0^{T_1}\sin\frac{2\pi nt}{T_1}\frac{Gm_1}{r_1(t)}\,dt,$$

$$b_n = \frac{2}{T_1}\int_0^{T_1}\cos\frac{2\pi nt}{T_1}V_1(t)\,dt = -\frac{2}{T_1}\int_0^{T_1}\cos\frac{2\pi nt}{T_1}\frac{Gm_1}{r_1(t)}\,dt.$$

In Fig. 194(b), we consider a stationary elliptic ring that has the shape of the orbit in Fig. 194(a). The mass per unit length of the ring is $\varrho(s)$. The gravitational potential of the ring at the point P is

$$V_{\text{ring}} = -\int_0^{S_1}\frac{G\varrho}{r_1(s)}\,ds,$$

where S_1 is the perimeter of the ellipse. The potential of the ring will

be equal to the constant term $\overline{V}_1$ of the Fourier series for the potential of the orbiting mass m_1 if

$$\varrho\, ds = \frac{m_1}{T_1}\, dt,$$

or

$$\frac{dm}{m_1} = \frac{dt}{T_1}, \tag{29}$$

where $dm = \varrho\, ds$ is the mass of the ring element with arc length ds, and dt is the time in which m_1 traverses ds. Equation (29) is equivalent to the statement of Gauss on page 634: the mass of the ring should be distributed so that equally large shares of the total mass are given to segments of the ellipse that are described in equally large times. According to Kepler's second law, the time dt required to traverse the arc length ds is proportional to the square of the distance between m_1 and the focus F. The mass per unit length of the elliptical ring is thus greatest at the point corresponding to the apogee of the elliptical orbit, as indicated by the line thickness in Fig. 194(b). If the orbit of m_1 is a circle in which the distance between m_1 and the focus F is constant, then the ring density ϱ is also constant.

Finding the ring density that corresponds to the nonconstant terms in the Fourier series for $V_1(t)$ is more difficult. To find the ring density that corresponds to the term with the time dependence $\sin 2\pi nt/T_1$, for example, we must find a density $\varrho(s,t)$ that satisfies the equation

$$\left(\frac{2}{T_1}\int_0^{T_1} \frac{Gm_1}{r_1(t)} \sin\frac{2\pi nt}{T_1}\, dt\right) \sin\frac{2\pi nt}{T_1} = \int_0^{S} \frac{G\varrho(s,t)}{r_1(s)}\, ds. \tag{30}$$

Solving equation (30) for $\varrho(s,t)$ is difficult, because the relation between s and t for an elliptical orbit is not simple. But for the case considered by Sommerfeld, in which the mass m_1 moves around F in a *circular* orbit, the relation between s and t is simply

$$\frac{s}{S} = \frac{t}{T_1}.$$

In this case, equation (30) is satisfied by the ring mass density

$$\varrho(s,t) = \frac{2m_1}{S} \sin\frac{2\pi ns}{S} \sin\frac{2\pi nt}{T_1}. \tag{31}$$

The spatial dependence of the ring density for the case $n = 2$ is illustrated by Sommerfeld on page 651. The density must also oscillate in time according to equation (31).

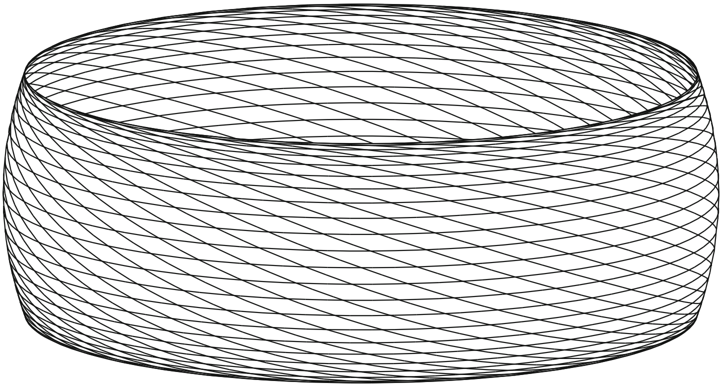

Fig. 195. Parallelogram elements of the Moon-ring.

227. (page 653) The parallelogram elements of the Moon-ring surface are illustrated in Fig. 195. The area dA of the element with angular coordinates (α, β) is

$$dA = r_2^2 \sin\phi \cos\alpha \, d\alpha \, d\beta,$$

where ϕ is the angle of inclination of the Moon's orbit with respect to the ecliptic. The mass per unit area of the element is therefore

$$\varrho = \frac{d\mu}{dA} = \frac{m_2}{4\pi^2 r_2^2 \sin\phi \cos\alpha},$$

which becomes infinite at the upper and lower edges ($\alpha = \pi/2$ and $\alpha = -\pi/2$, respectively) of the ring.

228. (page 662) Theodor Egon Ritter von Oppolzer (1841–1886) was an Austrian astronomer and mathematician. His most famous work is the *Canon der Finsternisse* [Oppolzer 1887], a catalogue of 8,000 solar and 5,200 lunar eclipses between 1207 B.C. and 2161 A.D.

229. (page 663) The Canadian-American astronomer and mathematician Simon Newcomb (1835–1909) became the director of the Nautical Almanac Office at the United States Naval Observatory in Washington, D.C., in 1877. In the preface to his *Elements of the Four Inner Planets and the Fundamental Constants of Astronomy* [Newcomb 1895], Newcomb writes that

> The diversity in the adopted values of the elements and constants of astronomy is productive of inconvenience to all

who are engaged in investigations based upon these quantities, and injurious to the precision and symmetry of much of our astronomical work. If any cases exist in which uniform and consistent values of all these quantities are embodied in an extended series of astronomical results, they are the exception rather than the rule. The longer this diversity continues the greater the difficulties which astronomers of the future will meet in utilizing the work of our time.

On taking charge of the work of preparing the *American Ephemeris* in 1877 the writer was so strongly impressed with the inconvenience arising from this source that he deemed it advisable to devote all the force which he could spare to the work of deriving improved values of the fundamental elements and embodying them in new tables of the celestial motion. It was expected that the work could all be done in ten years. But a number of circumstances, not necessary to describe at present, prevented the fulfillment of this hope. Only now is the work complete so far as regards the fundamental constants and the elements of the planets from Mercury to Jupiter inclusive. The construction of tables of the four inner planets is now in progress, those of Jupiter and Saturn having already been completed by Mr. Hill. All these tables will be published as soon as possible, and the investigations on which they are based are intended, so far as it is practicable to condense them, to appear in subsequent volumes of the *Astronomical Papers of the American Ephemeris*. As it will take several years to bring out these volumes, it has been deemed advisable to publish in advance the present brief summary of the work.

Newcomb's results were adopted by the 1896 *Conférence Internationale des Etoiles Fondamentales* in Paris, and his system became the international standard in 1901 [Carter 2009].

230. (page 663) According to the 1999 standards of the International Association of Geodesy, the principal moments of inertia of the Earth are

$$A = 8.0101 \pm 0.0002 \quad 10^{37}\ \text{kg m}^2,$$
$$B = 8.0103 \pm 0.0002 \quad 10^{37}\ \text{kg m}^2,$$
$$C = 8.0365 \pm 0.0002 \quad 10^{37}\ \text{kg m}^2,$$

so that

$$\frac{C-A}{C} = \frac{1}{304.4}, \quad \frac{C-A}{A} = \frac{1}{303.4}.$$

The currently accepted value for the ratio of the mass of the Earth to the mass of the Moon is 81.30.

231. (page 673) Christian August Friedrich Peters (1806–1880) was an assistant at the Pulkovo Observatory near St. Petersburg from 1839 to 1849; he was the editor of the *Astronomische Nachrichten* from 1854 until his death in 1880. The Swedish astronomer Magnus Nyrén (1837–1921) worked at Pulkovo from 1868 to 1907. James Clerk Maxwell (1831–1879) describes his search for the pole oscillations and the corresponding latitude variations at the end of his paper "On a Dynamical Top" [Maxwell 1890, pp. 260–261]:

> In order to determine the existence of such a variation of latitude, I have examined the observations of *Polaris* with the Greenwich Transit Circle in the years 1851–2–3–4. The observations of the upper transit during each month were collected, and the mean of each month found. The same was done for the lower transits. The difference of the zenith distance of upper and lower transit is twice the polar distance of Polaris, and half the sum gives the co-latitude of Greenwich.
>
> In this way I found the apparent co-latitude of Greenwich for each month of the four years specified.
>
> There appeared a very slight indication of a maximum belonging to the set of months,
>
> March 51. Feb. 52. Dec. 52. Nov. 53. Sept. 54.
>
> This result, however, is to be regarded as very doubtful, as there did not appear to be evidence for any variation exceeding half a second of space, and more observations would be required to establish the existence of so small a variation at all.
>
> I therefore conclude that the earth has been for a long time revolving about an axis very near to the axis of figure, if not coinciding with it. The cause of this near coincidence is either the original softness of the earth, or the present fluidity of its interior. The axes of the earth are so nearly equal, that a considerable elevation of a tract of country

might produce a deviation of the principal axis within the limits of observation, and the only cause which would restore the uniform motion, would be the action of a fluid which would gradually diminish the oscillations of latitude. The permanence of latitude essentially depends on the inequality of the earth's axes, for if they had been all equal, any alteration of the crust of the earth would have produced new principal axes, and the axis of rotation would travel about those axes, altering the latitudes of all places, and yet not in the least altering the position of the axis of rotation among the stars.

Perhaps by a more extensive search and analysis of the observations of different observatories, the nature of the periodic variation of latitude, if it exist, may be determined. I am not aware of any calculations having been made to prove its non-existence, although, on dynamical grounds, we have every reason to look for some very small variation having the period time of 325·6 days nearly, a period which is clearly distinguished from any other astronomical cycle, and therefore easily recognised.

232. (page 673) Karl Friedrich Küstner (1856–1936) worked at the observatories in Hamburg and Berlin before he was appointed to the chair of astronomy and the directorship of the university observatory in Bonn. Küstner's measurements that provided conclusive proof of a latitude variation in Berlin (by an amount of $0''.2$ over a period of one year from 1884 to 1885) are described in an article by the Bonn astronomer Peter Brosche [Brosche 2000].

233. (page 673) A biographical memoir of the American astronomer Seth Carlo Chandler (1846–1913) was published by the National Academy of Sciences in 1995 [Carter 1995]. The author of the memoir writes that the eighty-two-year delay between Chandler's death and the customary obituary notice by the Academy "is probably related to certain controversies in which Chandler became involved. Chandler's formal education reached only graduation from high school and he had virtually no theoretical background in astronomy or physics. However, he was a talented observer and an extraordinarily adroit computer, and he reported his observational and computational results with total disregard for conflicting accepted theory. As associate editor and later editor of the *Astronomical Journal*, Chandler had little difficulty

publishing and often included extensive commentaries in his technical papers. Chandler's comments undoubtedly proved particularly irritating to certain individuals simply because of his close association with Benjamin Peirce, B. A. [Benjamin Apthorp] Gould, and A. D. [Alexander Dallas] Bache. Just a few decades earlier these three scientists had joined forces in a highly publicized dispute over an attempt to develop a national observatory that ended in failure and left many personal animosities." The memoir cites an archive of Chandler's personal and professional correspondence, and gives a bibliography of his publications, including those cited by Sommerfeld on page 673.

234. (page 673) The 1891 expedition to Hawaii was led by the German astronomer Adolf Marcuse (1860–1930). Marcuse spent thirteen months in Hawaii, and later published his impressions of the islands and their inhabitants [Marcuse 1894].

Marcuse was met in Washington, D. C., and accompanied to Hawaii by Erasmus Darwin Preston (1851–1906) of the United States Coast and Geodetic Survey. Darwin's report on the expedition gives details of the many practical difficulties with rain, clouds, and unruly pack animals [Darwin 1893].

235. (page 675) A comprehensive history of the International Geodetic Association (*Die Internationale Erdmessung*), sponsor of the 1891 expedition to Hawaii and the parent organization of the Permanent Commission, is given by the former Association president Walter Davis Lambert (1879–1968) [Lambert 1950].

236. (page 675) Carl Theodor Albrecht (1843–1915) was the superintendent of the Department of Astronomical Geodesy in the Prussian Geodetic Institute. According to an obituary in the *Monthly Notices of the Royal Astronomical Society*, he was "of amiable character, and was active and vigorous, even at the advanced age of 71." Albrecht's reductions of the latitude observation data were published serially in the *Astronomische Nachrichten*, and were later compiled in the four-volume *Resultate des Internationale Breitendienstes* [Albrecht 1903–1911].

237. (page 679) Sir Franz Arthur Friedrich Schuster (1851–1934) was born in Frankfurt, Germany. He studied with Gustav Kirchhoff (1824–1887) and Robert Bunsen (1811–1899) in Heidelberg, and wrote a delightful memoir of his time there [Schuster 1932]. One of Schuster's fellow students in Heidelberg was Heike Kamerlingh Onnes, who in 1873 was already experimenting with a pendulum designed to demonstrate

the rotation of the Earth (see note 262). Schuster became professor of physics in Owens College, Manchester, and secretary of the Royal Society of London.

The paper cited by Sommerfeld [Schuster 1902] is an address by Schuster to the meeting of the British Association at Belfast. The address is devoted partly to Schuster's views on the organization of scientific meetings, and partly to a discussion of the periodgraph (or, now more commonly, periodgram), an analytic tool invented by Schuster for calculating the probability that an experimental time series contains a component with a specified period. Schuster compared his periodgraph with a spectroscope (the word was introduced by Schuster in 1882), which identifies the periodic components of light [Schuster 1900]. The computation of the periodgram has now become a standard part of discrete-time signal processing [Oppenheim 2010, pp. 836–849].

The 1,804 page report of Heinrich Burkhardt (1861–1914) on "Expansions in oscillatory functions and the integration of the differential equations of mathematical physics" [Burkhardt 1901] should be consulted by any person who considers writing a serious review article. The pages cited by Sommerfeld include a summary of Schuster's periodgraph.

238. (page 682) Hendrik Gerard van de Sande Bakhuyzen (1838–1923) was the director of the Leiden Observatory from 1872 to 1908. An obituary notice describing van de Sande Bakhuyzen's work in Leiden was written by the Dutch astronomer Willem de Sitter (1872–1934) [de Sitter 1924]. De Sitter became the director in Leiden in 1919 after the sudden death of Ernst Frederik van de Sande Bakhuyzen (1848–1918), who succeeded his brother Hendrik in 1908.

239. (page 684) Alexander Smith Christie (1846–1933) was an Irish immigrant who studied mathematics with Benjamin Peirce (1809–1880) at Harvard. In 1887, he was appointed head of the Tidal Division of the United States Coast and Geodetic Survey in Washington, D.C.

The Philosophical Society of Washington was founded in 1871 by the American scientist Joseph Henry (1797–1878), discoverer of electromagnetic self-inductance. Christie's article in the *Bulletin* of the Society [Christie 1895] contains a very detailed mathematical analysis of the tidal records for San Francisco, California, and Penobscot, Maine.

240. (page 684) The quotation is from *Der Spaziergang*, by Johann Christoph Friedrich von Schiller (1759–1805); the verse that contains

the quoted phrase is

Aber im stillen Gemach entwirft bedeutende Zirkel
  Sinnend der Weise, beschleicht forschend den schaffenden Geist,
Prüft der Stoffe Gewalt, der Magnete Hassen und Lieben,
  Folgt durch die Lüfte dem Klang, folgt durch den Aether dem Strahl;
Sucht das Vertraute Gesetz in des Zufalls grausenden Wundern,
  Sucht den ruhenden Pol in der Erscheinungen Flucht.

The poem was translated into English hexameters by the astronomer Sir John Frederick William Herschel (1792–1871), and published by the English scientist and philospher William Whewell (1794–1866). Herschel's translation of the given verse is [Whewell 1847, p. 19]

Science, the while, deep musing in cell over circle and figure,
  Knows and adores the Power which through creation it tracks,
Measures the forces of matter—the hates and loves of the magnets—
  Sound through its wafting breeze, Light through its Æther, pursues;
Seeks in the marvels of chance the law which pervades and controls it—
  Seeks the reposing pole, fixed in the whirl of events.

Herschel's views on the merits of the hexameter in English verse are expounded at some length in the preface to his translation of the entire *Iliad* of Homer [Herschel 1866].

241. (page 686) Sommerfeld's collaboration with the German geophysicist Emil Johann Wiechert (1861–1828) in the preparation of the present chapter is discussed in the preface to this volume.

242. (page 686) The English astronomer George Howard Darwin (1845–1912) was the son of the naturalist Charles Darwin (1809–1882). *The Tides and Kindred Phenomena in the Solar System* is based on a series of ten popular lectures that George Darwin gave in Boston, Massachusetts, in 1897 [Darwin 1898]. In Chapter XV of the book, Darwin discusses the implications of the Chandler nutation for the interior constitution of the Earth, and concludes that the interior of the Earth is "not improbably solid." He also mentions the possibility of deriving further indications concerning the physical conditions of the interior of the Earth from the detection and analysis of seismic waves. In the later chapters of the book, Darwin discusses "several branches of speculative Astronomy, with which the theory of the Tides has an intimate relationship," including the equilibrium figures of rotating fluids, the rings of Saturn, and the evolution of stars and planetary systems.

"The problems involved in the origin and history of the solar and of other celestial systems," writes Darwin, "have little bearing upon our life on Earth, yet these questions can hardly fail to be of interest to all those whose minds are in any degree permeated by the scientific spirit."

243. (page 686) The Swedish scientist Svante August Arrhenius (1859–1927) won the 1903 Nobel Prize in Chemistry for his electrolytic theory of dissociation. In the same year, he published his two-volume *Lehrbuch der kosmischen Physik*. Most of the work (half of Vol. 1 and all of Vol. 2) is devoted to the physics of the Earth. After discussing data for the increase of temperature with depth beneath the surface of the Earth, Arrhenius concludes that the Earth consists of a solid crust with a thickness of about 60 km, a layer of molten liquid magma with a thickness of about 240 km, and a gaseous core. The physical distinction between these three states of matter, however, is supposed not to be very great. The viscosity of the magma and the gaseous core are assumed to be very large because of the high pressure, and thus the magma and the core behave like solid bodies with respect to forces having the period of the Chandler nutation [Arrhenius 1903, p. 283].

244. (page 686) William Hopkins (1793–1866) studied and taught mathematics in the University of Cambridge. Aside from his investigation of the rotational properties of the supposedly fluid-filled Earth [Hopkins 1839, 1840, 1842], Hopkins is best known for having been the private tutor of Peter Tait (1831–1901), William Thomson (1824–1907), George Stokes (1819–1903), and James Clerk Maxwell (1831–1879).

245. (page 686) Lord Kelvin's (William Thomson's) conclusions about the dynamic properties of a fluid-filled shell are also given in the *Philosophical Transactions of the Royal Society of London* [Thomson 1863] and in Vol. I of the *Treatise on Natural Philosophy* [Thomson 1867, pp. 702–704].

246. (page 688) Newcomb's paper in the *Monthly Notices of the Royal Astronomical Society* gives a qualitative explanation of how the deformability of the Earth lengthens the period of the free nutation. He imagines viewing the Earth from above the North Pole, as we show in Figs. 196(a) and 196(b). In the free nutation of the rigid Earth (Fig. 196(a)), the rotation axis R rotates about the pole P that corresponds to the principal axis with the largest moment of inertia. The period of rotation is the Euler period T_E, and the velocity of R is $\omega_E l_{PR}$,

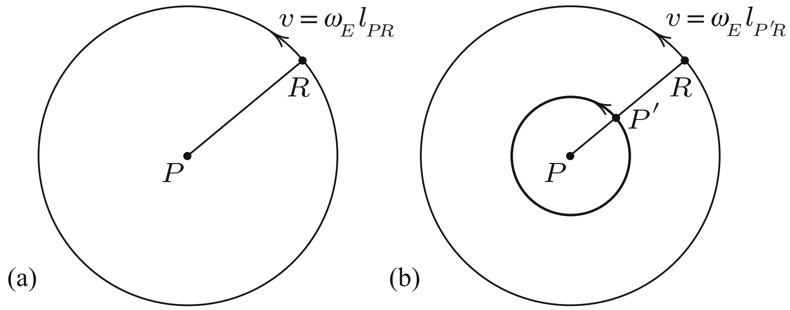

Fig. 196. Motion of the rotation axis R for the rigid and deformable Earth.

where $\omega_E = 2\pi/T_E$ and l_{PR} is the length of the radius PR. In the free nutation of the deformable Earth (Fig. 196(b)), the centrifugal bulge caused by the rotation around R displaces the principal axis with the largest moment of inertia to the point P'. The rotation axis R now rotates around P' with the Euler period while the point P' remains on line PR, so that P becomes the mean position of P'. The velocity of R is now $\omega_E l_{P'R}$, which is less than the velocity of R in Fig. 194(a). In Newcomb's words,

> The law of rotation of R is such that it constantly moves around the instantaneous position of P' in a period of 305 days, irrespective of the instantaneous motion of P' itself. In other words, the angular motion of R at each moment is that which it would have if P' had remained at rest. Hence, the angular motion as seen from P is less than that from P', in the ratio of $P'R : PR$.
>
> But, as R rotates, P' continually changes its position and rotates also, remaining on the straight line PR. Thus the time of revolution of R around P is increased in the same ratio.

Newcomb estimates the rigidity of the Earth that would be necessary, according to his argument, to change the nutation period from the Euler period to the Chandler period. He finds that the Earth must be slightly more rigid than steel.

The reader may decide whether Newcomb's argument is convincing. Be that as it may, the facts remain that Newcomb was the first to realize that the deformability of the Earth is the source of the difference

between the Euler period and the Chandler period, and that Newcomb's intuition was later confirmed by more detailed mathematical analyses.

247. (page 688) Sidney Samuel Hough (1870–1923) studied mathematics at St. John's College, Cambridge. He won the First Smith's Prize at Cambridge in 1894 for his paper entitled "The Oscillations of a Rotating Ellipsoidal Shell containing Fluid" [Hough 1895]. In this paper, Hough shows that the difference between the Euler period and the Chandler period cannot be explained by the presence of a fluid in the interior of the Earth, and concludes that Newcomb's hypothesis that the difference is due to the elastic compliability of the Earth is probably correct. In the 1896 paper cited by Sommerfeld [Hough 1896], Hough shows that the observed Chandler period corresponds to an elastic ellipsoid with a rigidity slightly less than that of steel.

According to an obituary by Sir Frank Watson Dyson, ninth Astronomer Royal, Hough worked as an astronomer at the Cape Observatory in South Africa, and in 1908 became the first president of the Royal Society of South Africa [Dyson 1923].

248. (page 689) Colin Maclaurin (1698–1746) became professor of mathematics in the University of Aberdeen at the age of nineteen, and later taught in the University of Edinburgh. Maclaurin's result for the ellipticity of a gravitating fluid mass that rotates as an ellipsoid of revolution is derived from the equations of hydrodynamics by Lamb [Lamb 1895, p. 583]; the ellipticity of the ellipsoid is not assumed to be small. (See also the references given by Sommerfeld in the footnote to p. 692.) Maclaurin himself did not have the hydrodynamic equations, which were introduced later by Euler. The oscillations and stability of the Maclaurin ellipsoid are discussed by Chandrasekhar in his *Ellipsoidal Figures of Equilibrium* [Chandrasekhar 1969, pp. 77–100].

249. (page 690) Sommerfeld's procedure here is questionable, since he previously expanded the potential V in a series that is valid when the distance r from the midpoint of the sphere to the external point P is large compared to the sphere radius R, and he now sets $r = R(1+\varepsilon(\cos^2\Theta - 2/3))$, where ε is small.

250. (page 691) Alexis Claude de Clairaut (1713–1765) published his *Théorie de la figure de la terre* in 1743. A detailed historical account of the work by Clairaut and his contemporaries on the problem of the figure of the Earth is given by Isaac Todhunter (1820–1884) [Todhunter 1873, Chs. VI–XI].

251. (page 702) Jean Charles Rodolphe Radau (1835–1911) was born in East Prussia, and studied mathematics at the University of Königsberg. He became a scientific journalist in Paris, and continued to work there on problems of mathematical astronomy. The fanciful side of Radau's character is revealed in the first chapters of his *L'Acoustique*, in which he tells marvelous tales of the bear in Brussels who played an organ consisting of twenty cats, shut in narrow boxes; of the nightingale who spoke Greek and Latin in the time of the emperor Claudius; and of the spider in Andria who jumped in time to the strains of a tarantella [Radau 1867].

Pierre Jean Octave Callandreau (1852–1904) was *Répétiteur* and later *Professeur* of astronomy at the *École Polytechnique* in Paris, and was an editor, with Rodolphe Radau, of the *Bulletin Astronomique.*

252. (page 707) Rudolf Ferdinand Spitaler (1849–1946) was professor of astrophysics in the Charles University of Prague. Spitaler's estimate of the yearly air mass displacement [Spitaler 1901] was later improved by the English geophysicist and mathematician Sir Harold Jeffreys (1891–1989). In addition to air mass displacements, Jeffreys considers the yearly mass transports due to the convection currents in the ocean, the formation of ice around the poles, the precipitation of both rain and snow, and the "periodical changes in vegetation, such as the formation of deciduous parts of trees, the rise of sap in them, and the formation of annual parts of herbs" (personal measurements are given for the densities of dead and live leaves in a beechwood on the Gagmagog Hills, Cambridge). Jeffreys concludes that "the known meteorological causes are apparently capable of giving a fairly good account of the observed annual motion of the pole, the errors found being perhaps within the range of uncertainty of the data" [Jeffreys 1916].

253. (page 708) Johannes Christian Lamp (1857–1891) was an astronomer at the Bothkamp Observatory in Schleswig-Holstein, and later in Berlin. In the paper cited by Sommerfeld, Lamp uses barometric data to estimate the seasonal elevation changes of the oceans. Lamp concludes that "the demonstrated displacement of water masses must have the greatest influence on the variability of the elevation of the pole" [Lamp 1891].

254. (page 708) Friedrich Robert Helmert (1843–1917) was professor of geodesy in the University of Berlin and director of the Geodetic Institute in Potsdam. The first volume (the mathematical part)

of his *Die Mathematischen und Physikalischen Theorieen der Höheren Geodäsie* was considered important enough to be translated into English in 1964 by the United States Air Force Aeronautical Chart and Information Center (ACIC), now part of the National Geospatial-Intelligence Agency (NGA) [Helmert 1880, 1964].

255. (page 709) The "inertia pole" is the principal axis with largest moment of inertia; it is the axis of revolution for the ellipsoid of inertia of an oblate axisymmetric body.

256. (page 715) Vito Volterra (1860–1940) was professor of mechanics in Pisa and Turin, and later professor of mathematical physics in Rome. Volterra's model for the explanation of the pole oscillations is shown in Fig. 197. A rigid body with center of mass at O has principal inertial axes ξ, η, ζ. The body contains an internal rigid and axisymmetric body PQ that rotates about an axis VV'. The axis VV' is fixed in $O\xi\eta\zeta$. The mass distribution of the body is completely unaffected by the rotation of PQ, but the dynamic behavior of the body can be quite strongly affected. Volterra considers the internal motion of PQ as an example of a "hidden motion" in the sense of Hertz, and also considers the system of Fig. 195 as a "cyclic system" of the type mentioned briefly by Klein and Sommerfeld on p. 224 of Vol. II. Volterra shows that the motion of a cyclic system can always be written in terms of Jacobi elliptic functions, and that the class of mechanical problems from which the Jacobi elliptic functions emerge is "much larger than that comprised in the classical investigations of Jacobi" [Volterra 1899, p. 210].

Giuseppe Peano (1858–1932) was professor of mathematics in the University and the Military Academy of Turin. Peano compares an internal motion of the Earth to the motion of a horse around a hippodrome. As the horse makes a circuit, the Earth must rotate in the opposite sense, but with the difference that when the horse has returned to its original position, the Earth has rotated through only a very small angle, and has assumed a different orientation [Peano 1895a, p. 515]. In a similar manner, as Peano remarks, a cyclic motion of the legs of a falling cat changes the orientation of the cat's body. Peano shows that an arbitrarily small internal motion that acts for a sufficiently long time can cause a secular displacement of the Earth's rotation axis, so that, for example, the present polar regions could be brought to the equator [Peano 1895b, p. 846]. Peano's papers are written in the language of his *calcolo geometrico*, a form of vector analysis that is based on the *Ausdehnungslehre* of Hermann Grassmann (cf. Vol. I, p. 259, note 97).

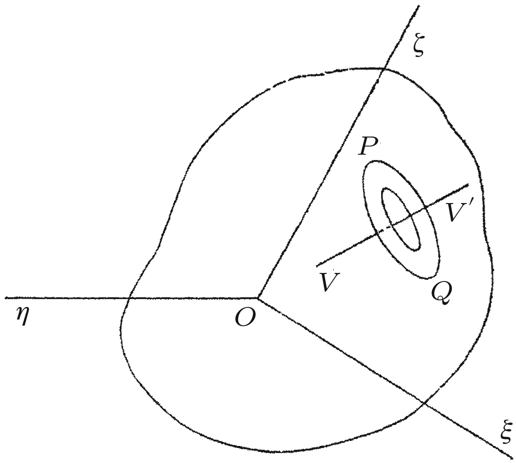

Fig. 197 Rigid body with internal cyclic motion [Volterra 1899, p. 203].

257. (page 715) Friedrich Heinrich Albert Wangerin (1844–1933) taught mathematics in the University of Berlin and the University of Halle-Wittenberg. The paper of Wangerin that is cited by Sommerfeld is summarized in the *Fortschritte der Physik* [Wangerin 1889]. Wangerin studies the motion of a rigid body that turns about a fixed point while carrying an axis about which a second body is free to turn. Various special cases concerning the properties of the two bodies are considered. Paul Rudolf Eugen Jahnke (1861–1921), a student of Wangerin who later taught at the *Bergakademie* and *Technische Hochschule* in Berlin, solves Wangerin's problem in terms of the Weierstrass sigma functions, and generalizes to the problem in which one body rotates about a fixed point while $n-1$ bodies rotate about axes that are fixed in the first body [Jahnke 1899]. At the end of his paper, Jahnke briefly considers the further generalization in which each of the $n-1$ bodies contains a series of m bodies, each of which rotates about an axis that is fixed in the preceding body.

258. (page 725) The 1754 paper of Immanuel Kant (1724–1804) on tidal friction is included in Vol. 8 of his *Werke* [Kant 1838, pp. 207–215]. After estimating the fluid mass and velocity corresponding to the tidal flow of the oceans, Kant states, without giving an explicit calculation, that the length of the day should be lengthened by a factor of two every two million years. An English translation of Kant's paper and a fascinating commentary on it are given by William Hastie (1842–1903),

professor of divinity in the University of Glasgow [Hastie 1900, pp. xxxix–liv, 3–11]. Kant claims that his paper was written in response to a prize question set by the Royal Academy of Science in Berlin. According to Hastie, however, there is no mention of any such prize question in the Transactions of the Academy, and no record of any other submissions for the prize.

259. (page 725) According to Darwin, the friction associated with the lunar tides will eventually cause the day (the period of the Earth's rotation with respect to the stars) to be equal to the month (the period of the Moon's rotation with respect to the Earth). The day and the month will each be equal to 55 of our current days, the Moon will always face the same part of the Earth's surface, and the lunar tides will cease. The friction associated with the solar tides will then increase the length of the day, while the length of the month will at first be unchanged. This will again generate lunar tides. The friction associated with these new lunar tides will be opposed to the friction of the solar tides, with the effect that Moon will slowly approach the Earth and must ultimately fall back into it.

Projecting into the past, Darwin concludes that there must have been a time when the day was equal to three to five of our present hours, the month was only slightly longer, and the Moon nearly touched the Earth's surface. Darwin speculates that this state of affairs occurred soon after the birth of the Moon, which was formed by a fission of the rapidly spinning Earth; Darwin first proposed this hypothesis for the formation of the Moon in 1878 [Darwin 1878]. The current view among planetary scientists is that Darwin's fission theory is incompatible with the present value of the angular momentum of the Earth–Moon system, and that the Moon was probably formed after a collision between the Earth and another body of the solar system [Canup 2004].

260. (page 731) Foucault's gyroscope is illustrated in Fig. 198. Foucault gives no detailed dimensions for his instrument, but states in the *Instructions sur les experériences du gyroscope* that the pointer attached to the outer ring is "30 to 40 centimeters long" [Foucault 1878, p. 417]. Detailed drawings of the rotor and the knife edges that support the inner ring are also given [Foucault 1878, plate 12]. A replica of Foucault's gyroscope is on display at the *Musée des arts et métiers* in Paris. According to William Tobin, author of a very fine book on Foucault and his work, the original instrument was bequeathed to the *Collège de France* and lost [Tobin 2003, p. 306].

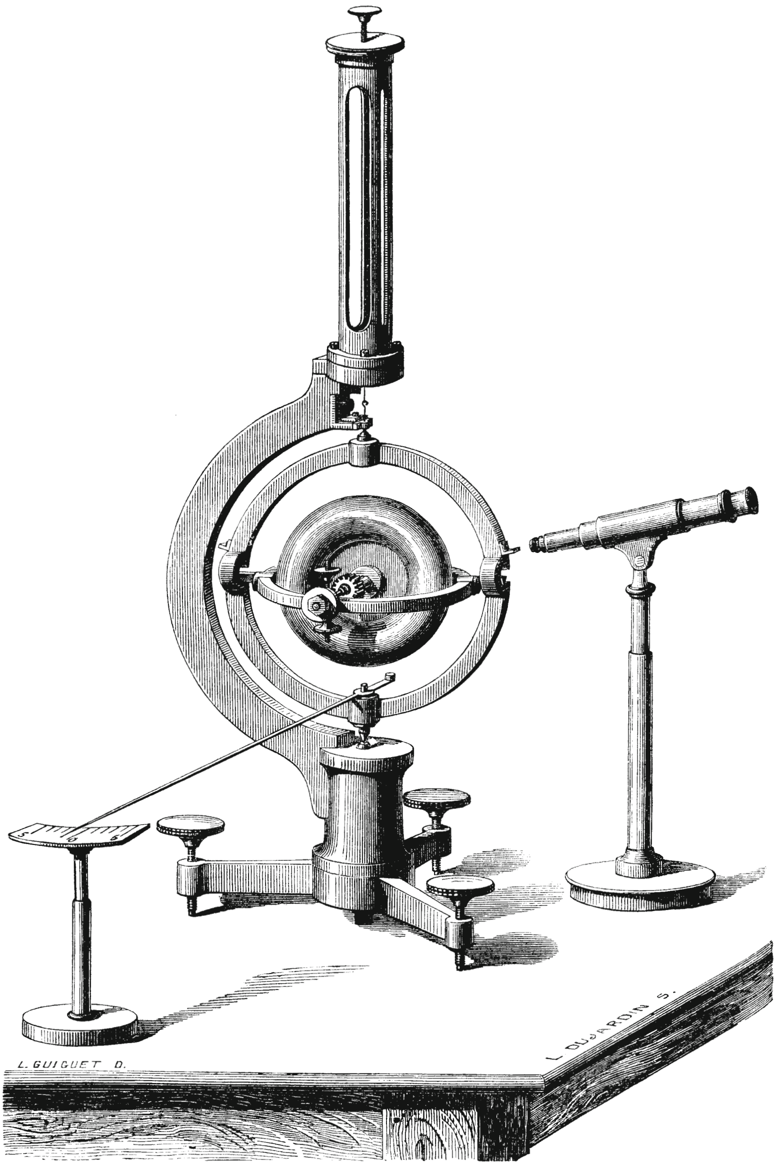

Fig. 198. Foucault's gyroscope for the experimental demonstration of the rotation of the Earth [Tobin 2003, p. 163].

261. (page 731) Foucault's papers in the *Comptes rendus* of the French *Académie des sciences* also appear in the posthumous *Recueil des travaux scientifiques de Léon Foucault* [Foucault 1878].

262. (page 732) Fig. 199 illustrates a gyroscope with three degrees of freedom on the surface of the Earth. In the position shown, the rotation axis of the outer ring is aligned with the local vertical, and the inner ring is parallel to the local horizon. The latitude of the gyroscope location is φ, and the rotor axis forms an angle ϑ with the local meridian. The rotor has angular velocity r with respect to its axis. If the Earth rotates with angular velocity ω and the axis of the rotor of the gyroscope is fixed in absolute space, then the angular velocity of the rotor with respect to the Earth has the components $\omega \sin \varphi$ with respect to the vertical, $\omega \cos \varphi \sin \vartheta$ with respect to the rotation axis of the inner ring, and $r + \omega \cos \varphi \cos \vartheta$ with respect to the axis of the rotor.

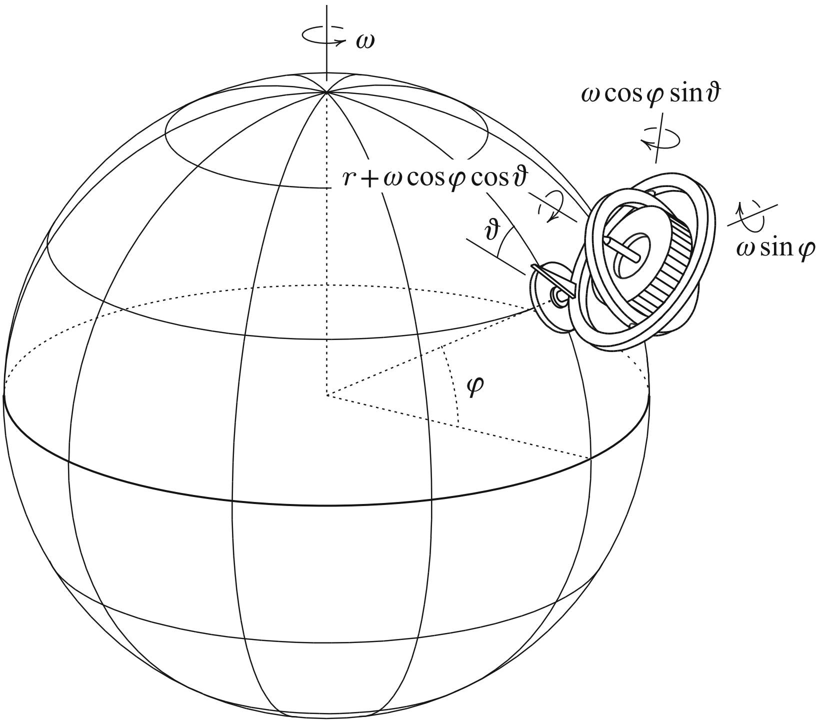

Fig. 199. Angular velocity components of the gyroscope rotor with respect to the rotating Earth.

263. (page 733) Heike Kamerlingh Onnes (1853–1926) was the discoverer of superconductivity and the winner of the 1913 Nobel Prize in Physics. In his doctoral work at the University of Groningen, Kamerlingh Onnes designed, built, and tested the first pendulum device that gave quantitative and conclusive experimental evidence of the rotation of the Earth [Kamerlingh Onnes 1879; Schulz-Dubois 1970]. In a comprehensive review of experimental proofs for the rotation of the Earth, the Jesuit astronomer Johann G. Hagen (1847–1930) writes that the design of Kamerlingh Onnes's pendulum is based in part on suggestions of the optician Ignazio Porro (1801–1875) and the mathematician Carl Friedrich Gauss (1777–1855) [Hagen 1911, pp. 74–75; Porro 1852]. Gauss became interested in the Foucault pendulum near the end of his life; a letter written at the age of seventy-six to Alexander von Humboldt contains the following passage [Gauss 1866b, p. 66]:

> I have recently occupied myself with the construction of an apparatus to accomplish the Foucault experiment in a different form. I have considered it a great fault of this experiment, as it has been executed by Foucault himself, by Secchi, and by others, that it requires a location that is rarely available. Secchi had, if I am not mistaken, an elevation of more than 100 feet [in the nave of Sant'Ignazio in Rome], Foucault an elevation of more than 200 feet [in the Panthéon in Paris], Garthe 134 feet [in the cathedral of Cologne], etc. My apparatus can be used at any location, and already now shows the effect of the Earth's rotation after a short time in the most striking manner. I hope, however (since it is still incomplete), to finish the still missing pieces, perhaps successively, so that everything appears with the greatest elegance and precision.

Porro makes a similar point about reducing the size of the Foucault pendulum; apparently both Porro and Gauss lacked Foucault's genius for showmanship and public appeal.

A physical model of the Gauss pendulum is mentioned by Emil Wiechert in a commemorative volume for the newly constructed buildings of the Physical Institute in Göttingen [Wiechert 1906, p. 130]:

> In the last years of his life, Gauss often devoted his attention to the problem of demonstrating the rotation of the Earth by means of the Foucault pendulum. Gauss himself published nothing on this subject, but we find many details

> in his letters. Above all, we possess in the collection of the Geophysical Institute a tangible testimony to his efforts on the Foucault pendulum, which was completed, according to an engraved inscription of 1853, two years before the death of GAUSS. The entire pendulum is in a cabinet of 2.45 m height, which is mounted directly on the floor. The Foucault thread suspension is replaced by a Cardanic knife-edge cross. A counterweight above the knife edges increases the period of the oscillation. To sharpen the observations, a mirror-reading is provided.
>
> Whether GAUSS worked with this instrument is not known; many imperfections indicate that it is no longer operative.— Independently of Gauss, KAMERLINGH ONNES constructed a very similar instrument, and achieved very good results with it; to reach this goal, many sources of error must certainly be eliminated with extraordinary care and labor.

The Gauss pendulum described by Wiechert is now lost. In 1997, however, a photograph of the instrument was found in the basement of the main building of the Geophysical Institute. The inscription of the photograph is written in English; it has been conjectured that the photograph was displayed as part of the Gauss–Weber exhibition at the 1893 World's Fair in Chicago [Siebert 1998].

264. (page 733) The October 7, 1854, edition of the London newspaper *The Athenæum* contains an article that describes Foucault's demonstration of his gyroscope at the twenty-fourth meeting of the British Association for the Advancement of Science. Foucault "spoke in French, but very distinctly, and the apparatus was so simple, beautiful, and exquisitely constructed that the experiments all succeeded to a miracle, and fully interpreted the author's meaning as he proceeded. . . . These beautiful and most decisive experiments were received most enthusiastically by the Section,—and at their close a request was made to the Committee of the Section to solicit the officers of the Association to have them repeated at one of the evening meetings before the assembled Association. This having been acceded to, they were repeated in the Great Hall of St. George's Hall on Tuesday evening, Dr. Tyndall interpreting M. Foucault's French as he proceeded; and on Wednesday they were again repeated in the Committee-room of Section A, to a select number of the *savans*."

265. (page 735) An inclination compass (also called a dip needle, a dip circle, or an inclinometer) is shown in Fig. 200. To use the instrument, the horizontal base circle is leveled and the upper circle is rotated about the pillar until the magnetic needle becomes vertical. The plane of the upper circle is then perpendicular to the plane of the local magnetic meridian. If the upper circle is rotated from this position by 90°, it will be parallel to the magnetic meridian, and the inclination of the needle will give the angle between the local magnetic field vector and the horizontal. If the magnetic field of the Earth were a pure dipole field, the inclination of the needle could be related to the local (magnetic) latitude.

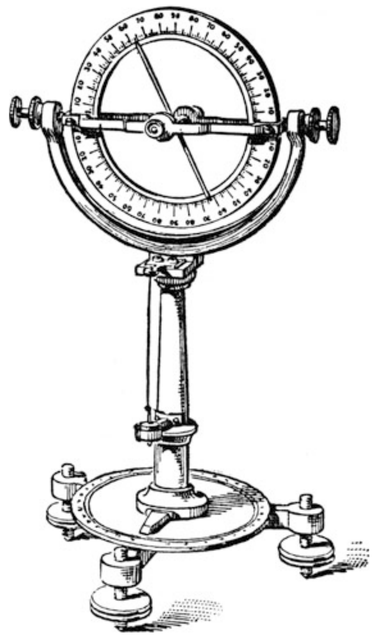

Fig. 200. Magnetic inclination compass [Whitney 1906, p. 1634].

266. (page 736) Jean Antoine Quet (1810–1884) was professor of physics in the Lycée Saint-Louis in Paris. Quet’s 1853 paper was cited by Foucault in a rebuttal to his critics; Foucault even claimed to have tested and verified some of the predictions in Quet’s paper [Quet 1853; Tobin 2003, p. 166].

Edmond Bour (1832–1866) was professor of mechanics in the *École des Mines* in Saint-Étienne and Paris. His 1863 paper on relative motion, first presented as a memoir to the *Académie des Sciences* in 1856, uses the analytic methods of Lagrange and Jacobi to derive the equations of motion for a mechanical system with respect to a reference frame that moves in an arbitrary manner.

267. (page 737) Émile Guyou (1843–1915) was a commandant in the French Navy and a teacher in the *École navale* in Brest. In the cited paper, Guyou considers the motion of the gyroscope with three degrees of freedom and no friction, and presents an obscure version of Sommerfeld’s argument on pp. 737–738.

268. (page 748) The gyroscope with two degrees of freedom on the rotating Earth is illustrated in Fig. 201. The axis of the rotor is constrained to move in a plane parallel to the local horizontal. The eigenimpulse of the rotor is N, the latitude of the gyroscope is φ, the angular velocity of the Earth is ω, and the angle between the rotor axis and the meridian is ϑ. If the angle ϑ were fixed, the rotor axis would move with respect to space on a cone with opening angle ϑ at the angular velocity $\omega\cos\varphi$. According to Sommerfeld's concept of the deviation resistance, the inertial moment $K = -N\omega\cos\varphi\sin\vartheta$ thus acts about the vertical to align the rotor axis with the meridian.

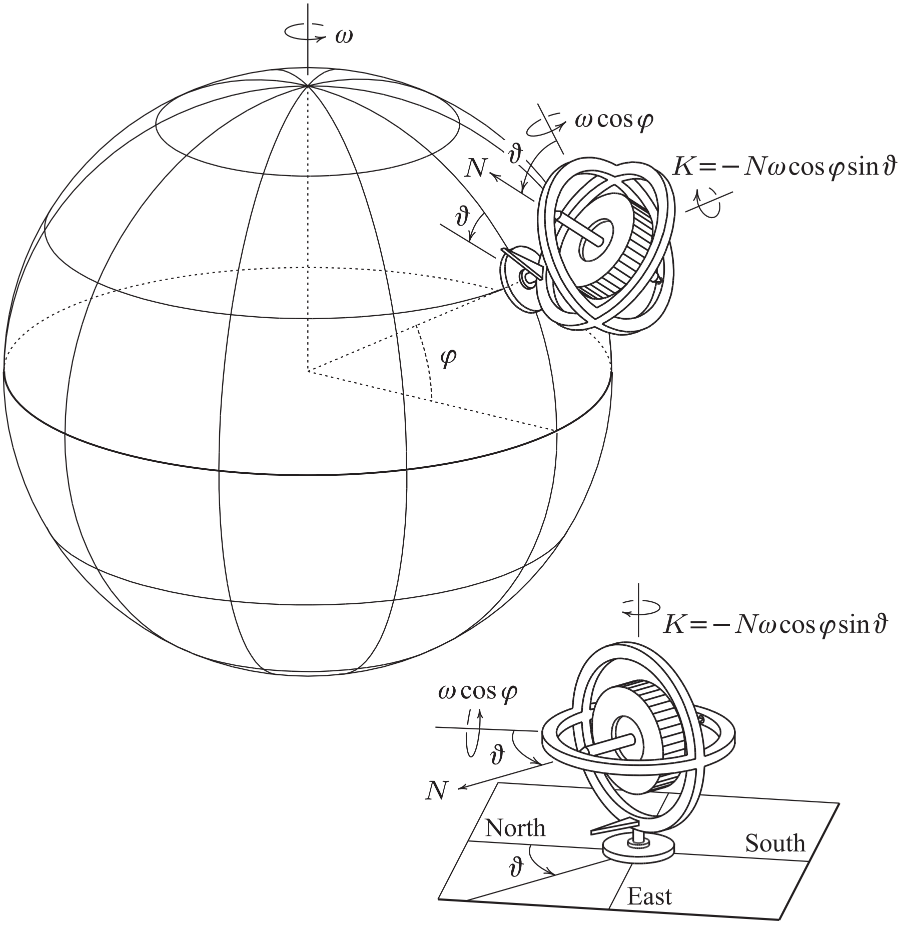

Fig. 201. Gyroscope with two degrees of freedom on the rotating Earth; rotor axis constrained to move in a horizontal plane.

269. (page 750) The gyroscope with two degrees of freedom on the rotating Earth is illustrated again in Fig. 202; the axis of the rotor is now constrained to move in the plane of the local meridian. The angle between the rotor axis and the Earth's rotation axis is ϑ. If the angle ϑ were fixed, the rotor axis would move with respect to space on a cone with opening angle ϑ at the angular velocity ω. According to Sommerfeld's concept of the deviation resistance, the inertial moment $K = -N\omega\cos\varphi\sin\vartheta$ thus acts about the axis of the outer ring to align the rotor axis with the direction of the Earth's axis.

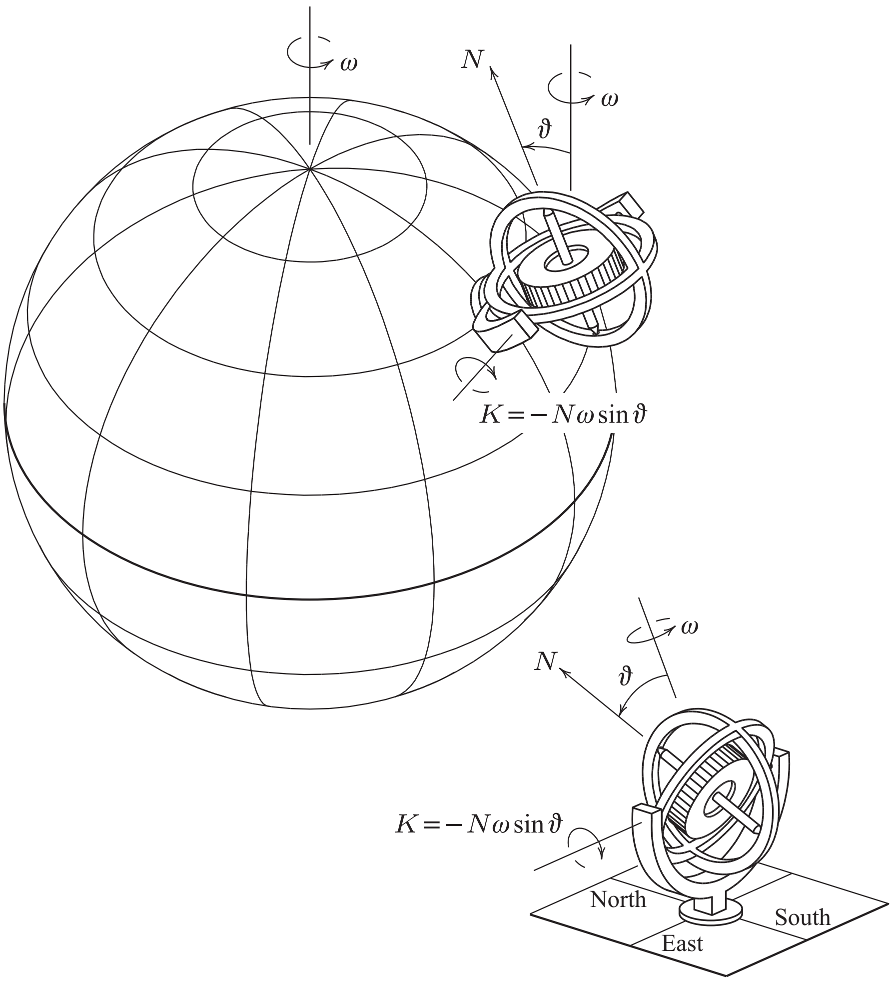

Fig. 202. Gyroscope with two degrees of freedom on the rotating Earth; rotor axis constrained to move in the meridional plane.

270. (page 755) The orientation angles for Gilbert's barogyroscope are illustrated in Fig. 203. The point O is at the center of the rotor. OV is the vertical, and OP is the direction of the rotation axis of the Earth, and therefore the (approximate) direction to the star Polaris. The plane of motion of the rotor axis forms the angle α with respect to the vertical plane through the north–south line. OQ is the projection of OP onto the plane of motion of the rotor axis. If the latitude of the barogyroscope location is φ, then the angle between OV and OP is $\pi/2 - \varphi$. The gravitational moment $M = -mg\delta \sin\chi$, where δ is the distance from the small mass m to O, acts to align the axis of the rotor with the vertical OV. The inertial moment $K = -N\omega \cos\lambda \sin\vartheta$, where N is the eigenimpulse of the rotor and ω is the angular velocity of the Earth, acts to align the axis of the rotor with the direction OQ.

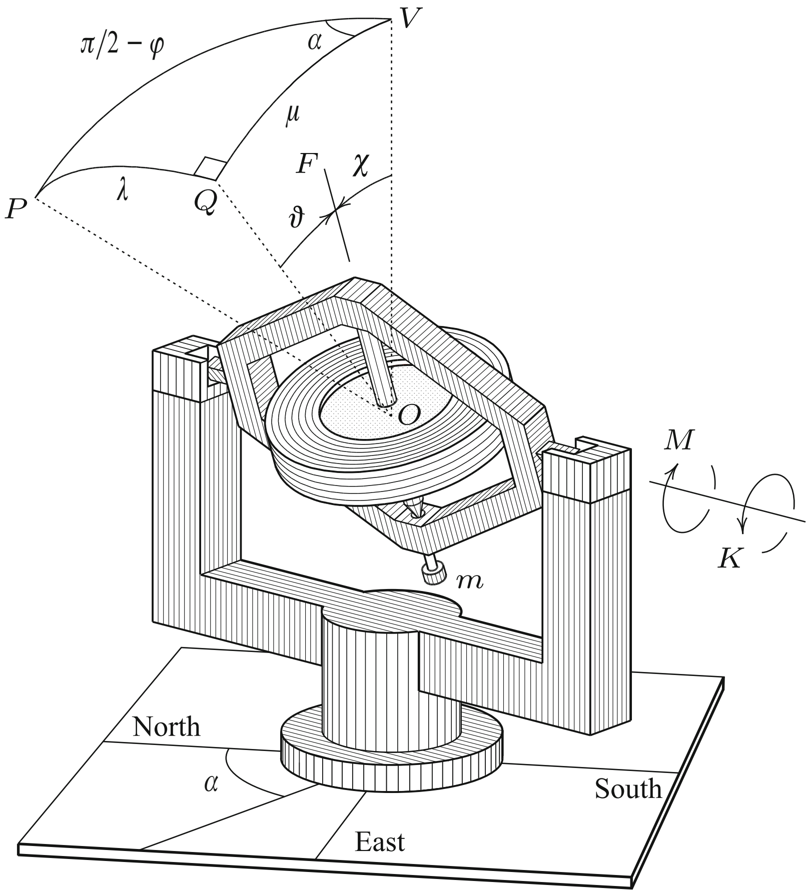

Fig. 203. Orientation angles for Gilbert's barogyroscope.

271. (page 758) In an 1883 paper in the *Journal de physique théorique et appliquée*, Gilbert states that his barogyroscope, "constructed with care and intelligence in the workshop of MM. E. Ducretet and Co., gives very clear results, perfectly conforming to the theory," but gives no experimental data of any kind [Gilbert 1883, p. 112]. See also the supplementary note by Klein and Sommerfeld on page 761.

272. (page 759) The first suggestion for an electrically driven gyroscope was made in 1851 by A. Krüger, professor in the gymnasium in Bromberg (now Bydgoszcz, Poland) [Krüger 1851]. Inspired by the Foucault pendulum, Krüger designed an electromagnet that rotates continuously in a complete circle. The rotating electromagnet is supported by a mobile frame, so that the plane of motion retains, in theory, a constant orientation in space while the Earth rotates beneath it. An attempt to build Krüger's apparatus is described by Caspar Garthe (1796–1876), professor of mathematics and natural science in the gymnasium in Cologne. Garthe gives no experimental results for his device, but presents it "with its defects in view, so that other artists may seek, where possible, to remove the mentioned difficulties in its construction, if they believe they are able to do so. The physical apparatus would then become a most highly useful instrument" [Garthe 1852, p. 35].

273. (page 761) These addenda and supplements by Klein and Sommerfeld were added when Vol. IV appeared in 1910.

274. (page 761) The original goal of Föppl's experiment was to detect a difference between the angular velocity of the Earth's rotation measured by the gyroscope and the same angular velocity measured by astronomical observations with respect to the fixed stars. Since the gyroscope presumably measures the angular velocity of the Earth with respect to absolute space, a difference in the two measured values would indicate a rotation of the fixed stars with respect to absolute space. Föppl writes that his original hope "was certainly not fulfilled" [Föppl 1904].

References.

[Albrecht 1903–1911]. Carl Theodor Albrecht and Bernhard Karl Wanach, *Resultate des internationalen Breitendienstes*, Georg Reimer, Berlin; Vol. I, 1903; Vol. II, 1906; Vol. III, 1909; Vol. IV, 1911; http://books.google.com.

[Arrhenius 1903]. Svante August Arrhenius, *Lehrbuch der kosmischen Physik*, S. Hirzel, Leipzig, 1903; http://books.google.com.

[Barus 1896]. Carl Barus, "A Curve-tracing Top," *Science*, Vol. IV, No. 91, 1896, pp. 444–446.

[Bevis 1754]. John Bevis, "*An historical Account and Description of Mr* SERSON'S *whirling* HORIZONTAL SPECULUM, *with its use in Navigation*," *Gentleman's Magazine*, Vol. XXIV, 1754, pp. 446–448; http://books.google.com.

[Bour 1863]. Edmond Bour, "*Mémoire sur les mouvements relatifs*," *Journal de mathématiques pures et appliquées*, 2e série, t. 8, 1863, pp. 1–51; http://gallica.bnf.fr.

[Brosche 2000]. Peter Brosche, "Küstner's Observations of 1884–85: the Turning Point in the Empirical Establishment of Polar Motion," *Polar Motion: Historical and Scientific Problems*, ASP Conference Series, Vol. 208, 2000, pp. 101–107; http://adsabs.harvard.edu.

[Burkhardt 1901]. Heinrich Burkhardt, "*Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*," *Jahresbericht der deutschen Mathematiker-Vereinigung*, Bd. 10, Heft 2, 1901, Erster Halbband, pp. 1–894, Zweiter Halbband, pp. 895–1804; http://gdz.sub.uni-goettingen.de.

[Canup 2004]. Robin M. Canup, "Dynamics of Lunar Formation," *Annual Review of Astronomy and Astrophysics*, Vol. 42, 2004, pp. 441–475.

F. Klein and A. Sommerfeld, *The Theory of the Top Volume III: Perturbations. Astronomical and Geophysical Applications*, DOI 10.1007/978-0-8176-4828-2,

[Carter 2009]. William E. Carter, "Simon Newcomb, America's first great astronomer," *Physics Today*, Vol. 62, 2009, pp. 46–51; http://scitation.aip.org.

[Chandrasekhar 1969]. Subrahmanyan Chandrasekhar, *Ellipsoidal Figures of Equilibrium*, Yale University Press, New Haven, 1969.

[Chandrasekhar 1995]. Subrahmanyan Chandrasekhar, *Newton's Principia for the Common Reader*, Clarendon Press, Oxford, 1995.

[Christie 1895]. Alexander Smith Christie, "The Latitude-Variation Tide," *Bulletin of the Philosophical Society of Washington*, Vol. 13, 1895–1899, pp. 103–122; http://books.google.com.

[Coulomb 1821]. Charles Augustin de Coulomb, *Théorie des machines simples*, Nouvelle édition, Bachelier, Paris, 1821; http://gallica.bnf.fr.

[Darwin 1878]. George Howard Darwin, "On the Precession of a Viscous Spheroid," *Nature*, Vol. XVIII, 1878, pp. 580–582; http://books.google.com.

[Darwin 1898]. George Howard Darwin, *The Tides and Kindred Phenomena in the Solar System*, Houghton, Mifflin and Co., Boston, 1898; http://books.google.com.

[Dyson 1923]. Frank Watson Dyson, "S. S. Hough," *The Observatory*, Vol. XLVI, 1923, pp. 268–272; http://adsabs.harvard.edu.

[Euler 1765]. Leonhard Euler, *Theoria motus corporum solidorum seu rigidorum*, A. F. Röse, Rostock and Greifswald, 1765; http://www.math.dartmouth.edu/~euler/.

[Föppl 1904]. August Föppl, "*Über einen Krieselversuch zur Messung der Umdrehungsgeschwindigkeit de Erde*," *Physikalische Zeitschrift*, Bd. 5, 1904, pp. 416–425; http://books.google.com.

[Foucault 1878]. Léon Foucault, *Recueil des travaux scientifiques de Léon Foucault*, Gauthier-Villars, Paris, 1878; http://books.google.com.

[Galton 1894]. Douglas Galton, *The Effects of Brakes upon Railway Trains*, new revised and enlarged edition, Westinghouse Air Brake Company, Pittsburgh, 1894; http://books.google.com.

[Garthe 1852]. Caspar Garthe, *Foucault's Versuch als direkter Beweis der Achsendrehung der Erde angestellt im Dom zu Köln, und erläutert durch zwei vorbereitende Vorlesungen, nebst Zusammenstellung einiger diesen Gegenstand betreffenden Apparate, Mittheilung wissenschaftlicher Versuchs-Reihen, und Beschreibung eines neuen Apparats, gennant Geostrophometer, mit welchem, ohne Pendel, die Achsendrehung der Erde erkannt werden kann*, Franz Carl Eisen, Köln, 1852; http://books.google.com.

[Gauss 1866a]. Carl F. Gauss, *Determinatio attractionis quam in punctum quoduis positionis datae exerceret planeta si eius massa per totam orbitam ratione temporis quo singulae partes describuntur uniformiter esset dispertita*, in *Werke*, Band 3, Dieterich, Göttingen, 1866; http://gdz.sub.uni-goettingen.de.

[Gauss 1866b]. *Briefe zwischen A. v. Humboldt und Gauss*, Wilhelm Engelmann, Leipzig, 1877; http://books.google.com.

[Gilbert 1883]. Philippe Gilbert, "*Sur les expériences propes a manifester la rotation de la terre*," *Journal de physique théorique et appliquée*, 2^e série, t. II, 1883, pp. 101–112; http://books.google.com.

[Guyou 1888]. Émile Guyou, "*Sur une solution élémentaire du problème du gyroscope de Foucault*," *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, t. 106, 1888, pp. 1143–1146; http://gallica.bnf.fr.

[Hagen 1911]. Johann G. Hagen, *La rotation de la terre ses preuves mécaniques anciennes et nouvelles*, Tipografia poliglotta Vaticana, Roma, 1911; http://adsabs.harvard.edu.

[Hastie 1839]. William Hastie, *Kant's Cosmogony*, James MacLehose and Sons, Glasgow, 1900; http://books.google.com.

[Helmert 1880, 1964]. Friedrich Robert Helmert, *Die mathematischen und physikalischen Theorieen der höheren Geodäsie*, Bd. I, B. G. Teubner, Leipzig, 1880; http://books.google.com; English transla-

tion *Mathematical and Physical Theories of Higher Geodesy*, Aeronautical Chart and Information Center, St. Louis, 1964; http://geographiclib.sourceforge.net.

[Herrmann 1882]. Gustav Herrmann, *Der Reibungswinkel*, Friedrich Vieweg und Sohn, Braunschweig, 1882.

[Herschel 1866]. John F. W. Herschel, *The Iliad of Homer, translated into English accentuated hexameters*, Macmillan and Co., London, 1866; http://books.google.com.

[Hopkins 1839]. William Hopkins, "Researches in Physical Geology. On the Phenomena of Precession and Nutation, assuming the Fluidity of the Interior of the Earth," *Philosophical Transactions of the Royal Society of London*, 1839, pp. 381–423; http://books.google.com.

[Hopkins 1840]. William Hopkins, "Researches in Physical Geology. On Precession and Nutation, assuming the Interior of the Earth to be fluid and heterogeneous," *Philosophical Transactions of the Royal Society of London*, 1840, pp. 193–208; http://books.google.com.

[Hopkins 1842]. William Hopkins, "Researches in Physical Geology. On the Thickness and Constitution of the Earth's Crust," *Philosophical Transactions of the Royal Society of London*, 1842, pp. 43–55; http://books.google.com.

[Hough 1895]. Sydney Samuel Hough, "The Oscillations of a Rotating Ellipsoidal Shell containing Fluid," *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 186, 1895, pp. 469–506; http://www.jstor.org.

[Hough 1895]. Sydney Samuel Hough, "The Rotation of an Elastic Spheroid," *Philosophical Transactions of the Royal Society of London*, Series A, Vol. 187, 1896, pp. 319–344; http://www.jstor.org.

[Jahnke 1899]. Eugen Jahnke, "*Nouvelles expressions des éléments d'un système orthogonal par les fonctions sigma d'un seul argument et leur application à la rotation de corps solides liés l'un à l'autre*," *Journal de mathématiques pures et appliquées*, sér. 5, t. 5, 1899, pp. 155–174; http://books.google.com.

[Jeffreys 1916]. Harold Jeffreys, "Causes contributory to the Annual Variation of Latitude," *Monthly Notices of the Royal Astronomical Society*, Vol. 76, 1916, pp. 499–525; http://adsabs.harvard.edu.

[Jellett 1872]. John Hewitt Jellett, *A Treatise on the Theory of Friction*, Hodges, Foster, and Co., Dublin, 1872; http://books.google.com.

[Jenkin 1877]. Fleeming Jenkin and James Alfred Ewing, "On Friction between Surfaces Moving at Low Speeds," *Philosophical Transactions of the Royal Society of London*, Vol. 167, 1877, pp. 509–528; http://www.-jstor.org.

[Kamerlingh Onnes 1879]. Heike Kamerlingh Onnes, *Nieuwe Bewijzen voor de Aswenteling der Aarde*, Groningen Univerisity thesis, 1879; http://gdz.sub.uni-goettingen.de.

[Kant 1839]. Immanuel Kant, "*Untersuchung der Frage, welche von der Königl. Akademie der Wissenschaften in Berlin zum Preise für das jetztlaufende Jahr aufgegeben worden: ob die Erde eine Veränderung ihrer Achsendrehung erlitten habe?*" in *Werke*, Bd. 8, Modes und Baumann, Leipzig, 1838; http://books.google.com.

[Klein 1899]. Felix Klein to Arnold Sommerfeld, Nov. 15, 1899, HS 1977-28/A,170, Deutsches Museum archive, Munich.

[Krüger 1851]. A. Krüger, "*Beschreibung eines Rotationapparates zur Demonstration der Axendrehung der Erde*," *Annalen der Physik und Chemie*, Bd. 84, 1851, pp. 151–152; http://books.google.com.

[Lamb 1895]. Horace Lamb, *Hydrodynamics*, University Press, Cambridge, 1895; http://books.google.com.

[Lambert 1950]. Walter Davis Lambert, "The International Geodetic Association (Die Internationale Erdmessing) and its Predecessors," *Bulletin Géodésique*, Vol. 17, 1950, pp. 299–324; http://adsabs.harvard.edu.

[Lamp 1891]. Johannes Christian Lamp, "*Ueber Niveauschwankungen der Oceane als eine mögliche Ursache der Veränderlichkeit der Polhöhe*," *Astronomische Nachrichten*, Vol. 126, 1891, pp. 223–224; http://adsabs.harvard.edu.

[Lindsay 1937]. Robert Bruce Lindsay, "Carl Barus (1856–1935)", *Proceedings of the American Academy of Arts and Sciences*, Vol. 71, No. 10, 1937, pp. 484–489; http://www.jstor.org.

[Lorenz 1902]. Hans Lorenz, *Lehrbuch der technischen Physik*, Oldenbourg, München and Berlin, 1902; http://books.google.com.

[Marcuse 1894]. Adolf Marcuse, *Die Hawaiischen Inseln*, R. Friedländer & Sohn, Berlin, 1894; http://books.google.com.

[Masi 1897]. Franceso Masi, *Le nouve vedute nelle ricerche teoriche ed esperimentali sull' attrito*, Nicola Zanichelli, Bologna, 1897; http://books.-google.com.

[Maxwell 1890]. James Clerk Maxwell, *The Scientific Papers of James Clerk Maxwell*, Vol. I, Cambridge University Press, 1890; http://books.-google.com.

[Morin 1832]. Arthur Jules Morin, *Nouvelles expériences sur le frottement, faites a Metz en 1831*, Bachelier, Paris, 1832; http://books.-google.com.

[Morin 1834]. Arthur Jules Morin, *Nouvelles expériences sur le frottement, faites a Metz en 1832*, Bachelier, Paris, 1834; http://books.-google.com.

[Morse 1953]. Philip M. Morse and Herman Feshbach, *Methods of Theoretical Physics*, Part II, McGraw-Hill, 1953.

[Mulligan 1892]. Joseph F. Mulligan, "Emil Wiechert (1861–1928): Esteemed seismologist, forgotten physicist," *American Journal of Physics*, Vol. 69, 2001, pp. 277–287; http://adsabs.harvard.edu.

[Newcomb 1892]. Simon Newcomb, "On the Dynamics of the Earth's Rotation, with respect to the Periodic Variations of Latitude," *Monthly Notices of the Royal Astronomical Society*, Vol. 52, 1982, pp. 366–341; http://scitation.aip.org.

[Newcomb 1895]. Simon Newcomb, *The Elements of the Four Inner Planets and the Fundamental Constants of Astronomy*, Government Printing Office, Washington, 1895; http://books.google.com.

[Oppenheim 2010]. Alan V. Oppenheim, *Discrete-Time Signal Processing*, Prentice Hall, 2010.

[Oppolzer 1887]. Theodor Ritter von Oppolzer, *Canon der Finsternisse*, Kaiserlich-Königlichen Hof- and Staatsdruckerei, Wien, 1887; http://www.archive.org.

[Peano 1895a]. Giuseppe Peano, "*Sopra lo spostamento del polo sulla terra*," *Atti della Reale Accademia delle Scienze di Torino*, Vol. XXX, 1895, pp. 515–523; http://books.google.com.

[Peano 1895b]. Giuseppe Peano, "*Sul moto del polo terrestre*," *Atti della Reale Accademia delle Scienze di Torino*, Vol. XXX, 1895, pp. 845–852; http://books.google.com.

[Perry 1907]. John Perry, *Applied Mechanics*, New revised and enlarged edition, D. Van Nostrand Co., New York, 1907; http://books.google.com.

[Porro 1852]. Ignazio Porro, "*La rotation de la Terre démontrée par le fixité du plan d'oscillation du pendule. Nouvel appareil pour l'observer*," *Comptes rendus hebdomadaires des séances de l'Académie des sciences*, t. 35, 1852, pp. 855–856; http://gallica.bnf.fr.

[Preston 1893]. Erasmus Darwin Preston, *On the Variations of Latitude and on Determinations of Gravity and the Magnetic Elements at Stations in the Hawaiian Islands*, United States Coast and Geodetic Survey, Appendix No. 13—Report for 1891, Government Printing Office, Washington, 1893; http://books.google.com.

[Quet 1853]. Jean Antoine Quet, "*Des mouvements relatifs en général, et spécialment mouvements relatifs sur la Terre*," *Journal de mathématiques pures et appliquées*, t. 18, 1853, pp. 213–298; http://gallica.bnf.fr.

[Radau 1867]. Rodolphe Radau, *L'acoustique, ou les phénomènes du son*, L. Hachette et C^ie^, Paris, 1867; http://books.google.com. English translation *Wonders of Acoustics; or, the Phenomena of Sound*, Scribner, Armstrong & Co., New York, 1872; http://books.google.com.

[Rennie 1829]. George Rennie, "Experiments on the Friction and Abrasion of the Surfaces of Solids," *Philosophical Transactions of the Royal Society of London*, Vol. 119, 1829, pp. 143–170; http://www.jstor.org.

[van de Sande Bakhuyzen 1894]. Hendrik Gerard van de Sande Bakhuyzen, "*Ueber die Aenderung der Polhöde*," *Astronomische Nachrichten*, Bd. 136, No. 3261, 1894, pp. 338–350; http://books.google.com.

[Schlömilch 1908]. Oskar Schlömilch, *Fünfstellige logarithmische und trigonometrische Tafeln*, 21st ed., Friedrich Vieweg und Sohn, Braunschweig, 1908; http://books.google.com.

[Schuster 1902]. Arthur Schuster, *Opening Address to the British Association at Belfast*, *Nature*, Vol. 66, No. 1720, 1902, pp. 614–618.

[Schuster 1932]. Arthur Schuster, *Biographical Fragments*, Macmillan and Co., London, 1932.

[Schuster 1900]. Arthur Schuster, "The Periodogram of Magnetic Declination as obtained from the records of the Greenwich Observatory during the years 1871–1895," *Transactions of the Cambridge Philosophical Society*, Vol. VXIII, 1900, pp. 107–135; http://books.google.com.

[Schwarzschild 1903]. Karl Schwarzschild to Arnold Sommerfeld, Mar. 29, 1903, HS 1977-28/A,318, Deutsches Museum archive, Munich.

[Short 1752]. James Short, "*An Account of an* horizontal Top, *invented by Mr.* Serſon," *Philosophical Transactions of the Royal Society of London*, Vol. XLVII, 1752, pp. 352–353; http://books.google.com.

[Siebert 1998]. Manfred Siebert, "*Das Foucault-Pendel von C. F. Gauß*," *Mitteilungen Gauss-Gesellschaft e.V. Göttingen*, 1998, pp. 49–52.

[de Sitter 1924]. Willem de Sitter, "Hendrik Gerard van de Sande Bakhuyzen," *Monthly Notices of the Royal Astronomical Society*, Vol. 84, 1924, pp. 226–230; http://adsabs.harvard.edu.

[Smith 1846]. Archibald Smith, "Note on the Theory of the Spinning-Top," *Cambridge Mathematical Journal*, Vol. 1, 1846, pp. 47–48; http://books.google.com.

[Sommerfeld 1902a]. Arnold Sommerfeld to Felix Klein, Feb. 15, 1902, Teubner archive, Stuttgart.

[Sommerfeld 1902b]. Arnold Sommerfeld to Felix Klein, Jun. 27, 1902, in Arnold Sommerfeld, *Wissenschaftlicher Briefwechsel*, Michael Eckert, ed., Vol. I, pp. 206–207, Deutsches Museum, Munich, 2000.

[Sommerfeld 1902c]. Arnold Sommerfeld, "*Zur Theorie der Eisenbahnbremsen*," *Denkschrift der Königlich Technischen Hochshule Aachen*, Aachener Verlag und Dresdner Gesellschaft, 1902, pp. 58–71; also in *Gesammelte Werke*, Friedr. Vieweg & Sohn, Braunschweig, 1968, Vol. I, pp. 432–445.

[Sommerfeld 1902d]. Arnold Sommerfeld to Karl Schwarzschild, Jul. 26, 1902, in Arnold Sommerfeld, *Wissenschaftlicher Briefwechsel*, Michael Eckert, ed., Vol. I, p. 208, Deutsches Museum, Munich, 2000.

[Sommerfeld 1902e]. Arnold Sommerfeld to Karl Schwarzschild, Aug. 12, 1902, in Arnold Sommerfeld, *Wissenschaftlicher Briefwechsel*, Michael Eckert, ed., Vol. I, pp. 208–210, Deutsches Museum, Munich, 2000.

[Sommerfeld 1903]. Arnold Sommerfeld to Karl Schwarzschild, Jan. 26, 1903, Niedersächsische Staats- und Universitätsbibliothek Göttingen archive, Schwarzschild 743.

[Sommerfeld 1904]. Arnold Sommerfeld, "*Zur hydrodynamischen Theorie der Schmiermittelreibung*," *Zeitschrift für Mathematik und Physik*, Vol. 50, 1904, pp. 97–155; also in *Gesammelte Werke*, Friedr. Vieweg & Sohn, Braunschweig, 1968, Vol. I, pp. 456–514.

[Sommerfeld 1916]. Arnold Sommerfeld, *"Die Quantentheorie der Spektrallinien und die letzte Arbeit von Karl Schwarzschild"*, *Die Umschau*, Vol. 20, 1916, pp. 941–946.

[Spitaler 1901]. Rudolf Spitaler, "*Die periodischen Luftmassenverschiebungen und ihr Einfluss auf die Lagenänderungen der Erdachse (Breitenschwankungen)*," *Dr. A. Petermanns Mitteilungen aus Justus Perthes geographischer Anstalt*, Supplementary Volume XXIX, No. 137, 1901; http://books.google.com.

[Stone 1867]. Edward James Stone, “On the Possibility of a Change in the Position of the Earth’s Axis due to Frictional Action connected with the Phenomena of the Tides,” *Monthly Notices of the Royal Astronomical Society*, Vol. XXVII, 1866/67, pp. 192–199; http://books.google.com.

[Thomson 1863]. William Thomson, “On the Rigidity of the Earth,” *Philosophical Transactions of the Royal Society of London*, Vol. 153, 1863, pp. 573–582; http://books.google.com.

[Thomson 1867]. William Thomson and Peter Guthrie Tait, *Treatise on Natural Philosophy*, Vol. I, Macmillan and Co., London, 1867; http://books.google.com.

[Tisserand 1889–1896]. François-Félix Tisserand, *Traité de mécanique céleste*, Gauthier-Villars, Paris, t. I, 1889, t. II, 1891, t. III, 1894, t. IV, 1896; http://www.archive.org.

[Tobin 2003]. William Tobin, *The Life and Science of Léon Foucault: The Man Who Proved the Earth Rotates*, Cambridge University Press, 2003.

[Todhunter 1873]. Isaac Todhunter, *A History of the Mathematical Theories of Attraction and the Figure of the Earth*, Vol. I, Macmillan and Co., London, 1873; http://books.google.com.

[Volterra 1899]. Vito Volterra, “*Sur la théorie des variations des latitudes*,” *Acta Mathematica*, Vol. 22, 1899, pp. 201–357; http://books.google.com.

[Wangerin 1895]. Albert Wangerin, “*Ueber die Rotation mit einander verbundener Körper*,” *Fortschritte der Physik*, 1895, pp. 311–314; http://books.google.com.

[Wegele 1882]. Franz X. von Wegele, *Geschichte der Universität Wirzburg*, Stahel’schen Buch- & Kunsthandlung, Wirzburg, 1882; http://books.google.com.

[Whewell 1847]. William Whewell, *English Hexameter Translations from Schiller, Göthe, Homer, Callinus, and Meleager*, John Murray, London, 1847; http://books.google.com.

[Whitney 1906]. William Dwight Whitney, ed., *The Century Dictionary*, Vol. II, Century Co., New York, 1906; http://books.google.com.

[Wiechert 1906]. Emil Wiechert, *Das Institut für Geophysik*, in *Die Physikalischen Institute der Universität Göttingen*, Teubner, Leipzig, 1906; http://books.google.com.

[Woolley 1963]. Sir Richard Woolley, "James Bradley, Third Royal Astronomer," *Quarterly Journal of the Royal Astronomical Society*, Vol. 4, 1963, pp. 47–52; http://adsabs.harvard.edu.

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